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Second Cohomology of the Modular Lie Superalgebra Ω[†]

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Abstract

We consider the finite-dimensional simple modular Lie superalgebra Ω which was defined by Zhang and Zhang (2009), over an algebraically closed fields of characteristic $p > 3$. In this paper, we determine the second cohomology group of the modular Lie superalgebra Ω by computing the first cohomology group $H^1(\Omega, \Omega^*)$, where Ω^* denotes the dual space of Ω.

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1. Introduction

Many important results of modular Lie superalgebras have been obtained (see, for example, [1, 5, 8]). But the classification problem is still open for the finite-dimensional simple modular Lie superalgebras. Since cohomology theory is closely related to the structures of modular Lie algebras and play an important role in the classification of modular Lie algebras(see [2, 3, 4]), it is significant to study the cohomology groups of modular Lie superalgebras. The dimensions of the second cohomology groups of simple modular Lie algebras of Cartan type were computed in [2, 3, 4]. The second cohomology groups of simple modular Lie superalgebras of Cartan type W, S, H and K were determined in [9, 11]. The second cohomologies of Lie Superalgebras HO and KO were studied in[6]. The second cohomologies of two classes of special odd modular Lie superalgebras were investigated in [7].

The finite-dimensional simple modular Lie superalgebra Ω was defined in [13]. Its derivation superalgebra and filtration structure were investigated in [12, 13]. The second cohomology group of modular Lie superalgebra Ω for $2n + 4 - q \not\equiv 0\,(\mathrm{mod} p)$, which possesses a nondegenerate associative form, can be easily obtained according to [9]. In

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paper [9], it was also verified that if a modular Lie superalgebra L was simple and did not possess any nondegenerate associative form, then its second cohomology group $H^2(L,\mathbb{F})$ was isomorphic to the first cohomology group $H^1(L,L^*)$. Thus we shall determine the second cohomology group of the modular Lie superalgebra Ω for $2n+4-q\equiv 0\,(\mathrm{mod} p)$, which does not have a nondegenerate associative form, by computing $H^1(\Omega,\Omega^*)$.

2. Preliminaries

Let $\mathbb{F}$ be an algebraically closed field of characteristic $p>3$ and not equal to its prime field Π. For $m>0$, let $E=\{z_1,\cdots,z_m\}\in\mathbb{F}$ be linearly independent over prime field Π and the additive subgroup H generated by E doesn't contain 1. If $\lambda\in H$, then we let $\lambda=\sum_{i=1}^{m}\lambda_i z_i$ and $y^{\lambda}=y_1^{\lambda_1}\cdots y_m^{\lambda_m}$, where $0\le\lambda_i<p$. We use the notation $\mathbb{N}$ for the set of positive integers and $\mathbb{N}_0$ for the set of non-negative integers. Let $\mathbb{Z}_2=\{\bar{0},\bar{1}\}$ be the ring of integers modulo 2.

Given $n\in\mathbb{N}$ and $r=2n+2$, we put $M=\{1,\cdots,r-1\}$. Suppose that $\mu_1,\cdots,\mu_{r-1}\in\mathbb{F}$ such that $\mu_1=0$, $\mu_j+\mu_{n+j}=1$, $j=2,\cdots,n+1$. Let $k_i\in\mathbb{N}_0$ for $i\in M$, then k_i can be uniquely expressed in p-adic form $k_i=\sum_{v=0}^{s_i}\varepsilon_v(k_i)p^v$, where $0\le\varepsilon_v(k_i)<p$. We define truncated polynomial algebra

$$A=\mathbb{F}[x_{10},x_{11},\cdots,x_{1s_1},\cdots,x_{r-10},x_{r-11},\cdots,x_{r-1s_{r-1}},y_1,\cdots,y_m]$$

such that

$$x_{ij}^p=0,\ \forall\, i\in M,\ j=0,1,\cdots,s_i;\ \ y_i^p=1,\ i=1,\cdots,m.$$

Let $Q=\{(k_1,\cdots,k_{r-1})\mid 0\le k_i\le\pi_i,\ \pi_i=p^{s_i+1}-1,\ i\in M\}$. If $k=(k_1,\cdots,k_r)\in Q$, we write $x^k=x_1^{k_1}\cdots x_{r-1}^{k_{r-1}}$, where ${x_i}^{k_i}=\prod_{v=0}^{s_i}x_{iv}^{\varepsilon_v(k_i)}$ for $i\in M$. For $0\le k_i,k_i'\le\pi_i$, it is easy to see that

$$(2.1)\qquad {x_i}^{k_i}{x_i}^{k_i'}={x_i}^{k_i+k_i'}\neq 0\Leftrightarrow\varepsilon_v(k_i)+\varepsilon_v(k_i')<p,\ v=0,1,\cdots,s_i.$$

Let $\Lambda(q)$ be the Grassmann superalgebra over $\mathbb{F}$ in q variables $\xi_{r+1},\cdots,\xi_{r+q}$ with $q\in\mathbb{N}$ and $q>1$. Denote the tensor product by $\widetilde{\Omega}:=A\otimes_{\mathbb{F}}\Lambda(q)$. Obviously, $\widetilde{\Omega}$ is an associative superalgebra with a $\mathbb{Z}_2$-gradation induced by the trivial $\mathbb{Z}_2$-gradation of A and the natural $\mathbb{Z}_2$-gradation of $\Lambda(q)$:

$$\widetilde{\Omega}_{\bar{0}}=A\otimes_{\mathbb{F}}\Lambda(q)_{\bar{0}},\quad \widetilde{\Omega}_{\bar{1}}=A\otimes_{\mathbb{F}}\Lambda(q)_{\bar{1}}.$$

For $f\in A$ and $g\in\Lambda(q)$, we abbreviate $f\otimes g$ to fg. For $k\in\{1,\cdots,q\}$, we set

$$\mathbb{B}_k=\{\langle i_1,i_2,\cdots,i_k\rangle\mid r+1\le i_1<i_2<\cdots<i_k\le r+q\}$$

and $\mathbb{B}(q)=\bigcup_{k=0}^{q}\mathbb{B}_k$, where $\mathbb{B}_0=\varnothing$. If $u=\langle i_1,\cdots,i_k\rangle\in\mathbb{B}_k$, we let $|u|=k,\{u\}=\{i_1,\cdots,i_k\}$ and $\xi^u=\xi_{i_1}\cdots\xi_{i_k}$. Put $|\varnothing|=0$ and $\xi^{\varnothing}=1$. Then $\{x^k y^{\lambda}\xi^u\mid k\in Q,\lambda\in H,u\in\mathbb{B}(q)\}$ is an $\mathbb{F}$-basis of $\widetilde{\Omega}$.

If L is a Lie superalgebra, then $h(L)$ denotes the set of all $\mathbb{Z}_2$-homogeneous elements of L, i.e., $h(L)=L_{\bar{0}}\cup L_{\bar{1}}$. If $|x|$ appears in some expression in this paper, we always regard x as a $\mathbb{Z}_2$-homogeneous element and $|x|$ as its $\mathbb{Z}_2$-degree.

Set $s=r+q$, $T=\{r+1,\cdots,s\}$ and $R=M\cup T$. Put $M_1=\{2,\cdots,r-1\}$. Define $\tilde{i}=\bar{0}$, if $i\in M_1$, and $\tilde{i}=\bar{1}$, if $i\in T$. Let

$$i'=\begin{cases} i+n, & 2\le i\le n+1,\\ i-n, & n+2\le i\le r-1,\\ i, & r+1\le i\le s,\end{cases}\qquad [i]=\begin{cases} 1, & 2\le i\le n+1,\\ -1, & n+2\le i\le r-1,\\ 1, & r+1\le i\le s.\end{cases}$$

For $e_i = (\delta_{i0}, \cdots, \delta_{ir})$, $i \in M$, we abbreviate x^{e_i} to x_i. Let D_i, $i \in R$, be the linear transformation of $\widetilde{\Omega}$ such that

$$D_i(x^k y^\lambda \xi^u) = \begin{cases} k_i^* x^{k-e_i} y^\lambda \xi^u, & i \in M, \\ x^k y^\lambda \cdot \partial \xi^u / \partial \xi_i, & i \in T, \end{cases}$$

where k_i^* is the first nonzero number of $\varepsilon_0(k_i), \varepsilon_1(k_i), \cdots, \varepsilon_{s_i}(k_i)$. Then $D_i \in \operatorname{Der} \widetilde{\Omega}$. Set

$$\bar{\partial} = I - \sum_{j \in M_1} \mu_j x_{j\,0} \frac{\partial}{\partial x_{j\,0}} - \sum_{j=1}^{m} z_j y_j \frac{\partial}{\partial y_j} - 2^{-1} \sum_{j \in T} \xi_j \frac{\partial}{\partial \xi_j},$$

where I is the identity mapping of $\widetilde{\Omega}$. For $f \in h(\widetilde{\Omega})$ and $g \in \widetilde{\Omega}$, we define a bilinear operation $[\ ,\ ]$ in $\widetilde{\Omega}$ such that

$$[f, g] = D_1(f)\bar{\partial}(g) - \bar{\partial}(f) D_1(g) + \sum_{i \in M_1 \cup T} [i](-1)^{\tilde{i}|f|} D_i(f) D_{i'}(g). \tag{2.2}$$

Then $\widetilde{\Omega}$ becomes a Lie superalgebra for the operation $[\ ,\ ]$ defined above.

Define $\Omega := [\widetilde{\Omega}, \widetilde{\Omega}]$. It can be proved that $\Omega = \operatorname{span}_{\mathbb{F}}\{x^k y^\lambda \xi^u | (k, \lambda, u) \neq (\pi, 0, \omega)\}$ for $2n + 4 - q \equiv 0 \,(\operatorname{mod} p)$, $\pi = (\pi_1, \cdots, \pi_{r-1})$ and $\omega = \langle r+1, \cdots, s\rangle$, and Ω is a simple Lie superalgebra. In the sequel, we always assume that $2n + 4 - q \equiv 0 \,(\operatorname{mod} p)$.

Now we give a $\mathbb{Z}$-gradation of Ω: $\Omega = \oplus_{j \in X} \Omega_j$, where

$$\Omega_j = \operatorname{span}_{\mathbb{F}}\{x^k y^\lambda \xi^u \mid \sum_{i \in M_1} k_i + 2k_1 + |u| - 2 = j\}, \tag{2.3}$$

and $X = \{-2, -1, \cdots, \tau\}$, $\tau = \sum_{i \in M_1} \pi_i + 2\pi_1 + q - 2$. If $f \in \Omega_j$, then f is called a $\mathbb{Z}$-homogeneous element and j is the $\mathbb{Z}$-degree of f which is denoted by $\operatorname{zd}(f)$.

Assume that $L = L_{\bar{0}} \oplus L_{\bar{1}}$ is a finite-dimensional modular Lie superalgebra and L possesses a $\mathbb{Z}$-gradation $L = \oplus_{i=-\varsigma}^{\delta} L_i$. Then $L^* := \operatorname{Hom}_{\mathbb{F}}(L, \mathbb{F}) = \oplus_{i=-\delta}^{\varsigma} (L^*)_i$ is a $\mathbb{Z}$-gradation L-module by virtue of $(x \cdot f)(y) = -(-1)^{|x||f|} f([x, y])$ for $x, y \in L$, $f \in L^*$.

Let $\bar{H} \subset L_0 \cap L_{\bar{0}}$ be a nilpotent subalgebra of $L_{\bar{0}}$. Let

$$L = \oplus_{\alpha \in \Delta} L_{(\alpha)} \quad \text{and} \quad L^* = \oplus_{\beta \in \Theta} (L^*)_{(\beta)}$$

be the weight space decompositions of L and L^* with respect to $\bar{H}$, respectively. Since $\bar{H} \subset L_0 \cap L_{\bar{0}}$, there exist subsets $\Delta_i \subset \Delta$ and $\Theta_j \subset \Theta$ such that

$$L_i = \oplus_{\alpha \in \Delta_i} L_i \cap L_{(\alpha)} \quad \text{and} \quad (L^*)_j = \oplus_{\beta \in \Theta_j} (L^*)_j \cap (L^*)_{(\beta)}.$$

Thus L has a structure of ($\mathbb{Z}\times$ $\operatorname{Map}(\bar{H}, \mathbb{F})$)-gradation, where $\operatorname{Map}(\bar{H}, \mathbb{F})$ is the group consisting of the mappings from $\bar{H}$ into $\mathbb{F}$. The L-module L^* is ($\mathbb{Z} \times \operatorname{Map}(\bar{H}, \mathbb{F})$)-graded by

$$(L^*)_{(i,\alpha)} = \{f \in L^* \mid f(L_j \cap L_{(\beta)}) = 0,\ (j, \beta) \neq -(i, \alpha)\}.$$

Then we have $(L^*)_{(i,\alpha)} = (L^*)_i \cap (L^*)_{(\alpha)}$.

By equation (2.2), $\bar{H} := \mathbb{F}x_1$ is an Abelian subalgebra of Ω. Furthermore, $\bar{H}$ is an Abelian subalgebra of $\Omega_0 \cap \Omega_{\bar{0}}$ with the weight space decomposition $\Omega = \oplus_{\alpha \in \Delta} \Omega_{(\alpha)}$. It is easy to see that $\Delta_i \cap \Delta_j \neq \emptyset$ if and only if $i \equiv j \,(\operatorname{mod} p)$.

2.1. Lemma. [11] *Let $L^* = \oplus_{\beta \in \Theta} (L^*)_{(\beta)}$ be the weight space decomposition relative to $\bar{H}$. Then the following statements hold:*
(1) $\Theta = -\Delta$ *and there is an isomorphism* $(L^*)_{(\beta)} \cong (L_{(-\beta)})^*$ *of $\bar{H}$-modules for all* $\beta \in \Theta$.
(2) $\Theta_i = -\Delta_{-i}$ *for* $-\delta \leq i \leq \varsigma$.

2.2. Definition. A linear mapping $\psi : L \to L^*$ is called a derivation if

$$\psi([x, y]) = (-1)^{|\psi||x|} x \cdot \psi(y) - (-1)^{|\psi(x)||y|} y \cdot \psi(x) \quad \text{for all } x, y \in L.$$

Let $\mathrm{Der}_{\mathbb{F}}(L, L^*)$ denote the space of derivations from L into L^* and $\mathrm{Inn}_{\mathbb{F}}(L, L^*)$ be the subspace of inner derivations. Recall that a derivation ψ from L into L^* is called inner if there is some $f \in L$ such that

$$\psi(x) = -(-1)^{|f||x|}x \cdot f \quad \text{for all } x \in L.$$

$\mathrm{Der}_{\mathbb{F}}(L, L^*)$ inherits the $(\mathbb{Z}\times\mathrm{Map}(\bar{H}, \mathbb{F}))$-gradation from L and L^*. A derivation $\psi \in \mathrm{Der}_{\mathbb{F}}(L, L^*)$ is referred to as homogeneous of degree (i_0, α_0) provided that

$$\psi(L_i \cap L_{(\alpha)}) \subset (L^*)_{i+i_0} \cap (L^*)_{(\alpha+\alpha_0)} \quad \text{for all } (i, \alpha) \in \mathbb{Z} \times \mathrm{Map}(\bar{H}, \mathbb{F}).$$

Let $\{f_1, \cdots, f_m\}$ be an $\mathbb{F}$-basis of $L_{\bar{0}}$ and $\{g_1, \cdots, g_n\}$ be an $\mathbb{F}$-basis of $L_{\bar{1}}$. Let $U(L)$ denote the universal enveloping algebra of L and $L^- = \sum_{i=-\varsigma}^{-1} L_i$. As the $U(L)$-module structure of L is induced by the L-module structure of L, we see that

$$(2.4) \qquad (f_1^{s_1} \cdots f_m^{s_m} g_{i_1} \cdots g_{i_t}) \cdot z = (\mathrm{ad} f_1)^{s_1} \cdots (\mathrm{ad} f_m)^{s_m} \mathrm{ad} g_{i_1} \cdots \mathrm{ad} g_{i_t}(z),$$

where $\{f_1^{s_1} \cdots f_m^{s_m} g_{i_1} \cdots g_{i_t} | s_i \geq 0,\ i = 1, \cdots, m;\ 1 \leq i_1 \leq \cdots \leq i_t \leq n\}$ is an $\mathbb{F}$-basis of $U(L)$.

2.3. Lemma. [11] *Suppose that $L = U(L^-) \cdot L_\delta$ and $\psi : L \to L^*$ is a homogeneous linear mapping of degree $l > 2\varsigma - \delta$. Assume that L^- is generated by a subset J of L. If*

$$\psi([x, y]) = (-1)^{|\psi||x|}x \cdot \psi(y) - (-1)^{|\psi(x)||y|}y \cdot \psi(x) \quad \text{for all } x \in J \text{ and } y \in L,$$

then ψ is a derivation.

2.4. Definition. A derivation $\psi : L \to L^*$ is said to be skew if

$$\psi(x)(y) = -(-1)^{|x||y|}\psi(y)(x) \quad \text{for all } x, y \in L.$$

Let $U(L)^+$ denote the two-sided ideal generated by L. Clearly, for every derivation $\psi : L \to L^*$, there exists a homomorphism $\varphi : U(L)^+ \to L^*$ of $U(L)$-modules such that $\varphi(x) = \psi(x)$ for all $x \in L$.

2.5. Lemma. [11] *Let $\psi : L \to L^*$ be a derivation. Assume $e \in L$ such that $(\mathrm{ad} e)^{p^t} = 0$ for $t \in \mathbb{N}$. Then $e^{p^t-1} \cdot \psi(e) \in (L^*)^L$, where*

$$(L^*)^L = \{f \in L^* \mid L \cdot f = 0\} = \{f \in L^* \mid f([L, L]) = 0\}.$$

2.6. Lemma. [11] *Let $V \subset L$ be a $\mathbb{Z}_2$-graded subspace such that*

$$L = U(L^-)^+ \cdot V \oplus V.$$

Let $W \subset \mathbb{N}_0^n$ and $\{e_1, e_2, \cdots, e_n\}$ be a basis of L^- such that
(a) $\mathrm{ann}_{U(L^-)^+}(L) = \mathrm{span}_{\mathbb{F}}\{e^b \mid b \notin W\}$, where $b = (b_1, b_2, \cdots, b_n)$ and $e^b := e_1^{b_1} e_2^{b_2} \cdots e_n^{b_n}$, and $\mathrm{ann}_{U(L^-)^+} := \{u \in U(L^-)^+ \mid u \cdot L = 0\}$;
(b) there is a basis $\{v_1, v_2, \cdots, v_m\}$ of V such that $\{e^a \cdot v_j \mid a \in W, 1 \leq j \leq m\}$ is a basis of L over $\mathbb{F}$.
Then the following statements hold:
(1) If $\mu_i = p^{k_i} - 1$ for $1 \leq i \leq n$ and $L = [L, L]$, then the canonical mapping $\Phi_1 : H^1(L, L^*) \to H^1(L^-, L^*)$ is trivial.
(2) If $\psi : L \to L^*$ is a derivation satisfying $\ker(\mathrm{ad} e_i) \subset \ker\psi(e_i)$ for $1 \leq i \leq n$, then ψ defines an element of $\ker\Phi_1$.
(3) If there is a $\mu \in \mathbb{N}_0^n$ such that $W = \{b \in \mathbb{N}_0^n \mid b \leq \mu\}$, then $\ker(\mathrm{ad} e_i) \subset \ker\psi(e_i)$ if and only if $e_i^{\mu_i} \cdot \psi(e_i) = 0$.

2.7. Lemma. [11] *Suppose $L = \oplus_{i=-\varsigma}^{\delta} L_i$. Let $\psi : L \longrightarrow L^*$ be a homogeneous derivation of degree l.*
(1) If $l > \varsigma - \delta$ and ψ defines an element of $\ker\Phi_1$, then ψ is an inner derivation.
(2) If $l = \varsigma - \delta$, ψ is skew and defines an element of $\ker\Phi_1$, then ψ is an inner derivation.

2.8. Lemma. [11] *Suppose* $L = \oplus_{i=-\varsigma}^{\delta} L_i$. *Let* $\psi : L \longrightarrow L^*$ *be a homogeneous derivation of degree* l *with* $-2\delta \le l \le -\delta - 1$. *If* $-\Delta_\delta \not\subset \phi_{-(\delta+l)}$, *then* $\psi = 0$, *where* $\phi_d \subset \Delta_d$ *for* $d > 1$.

3. Second cohomology group $H^2(\Omega, \mathbb{F})$

3.1. Proposition. *Suppose that* $\psi : \Omega \to \Omega^*$ *is a skew derivation of degree* $l \ge 5 - \tau$. *Then there exists a homogeneous skew derivation* $\widetilde{\psi} : \widetilde{\Omega} \to \widetilde{\Omega}^*$ *of degree* l *which extends* ψ.

Proof. We define a linear mapping $\widetilde{\psi} : \widetilde{\Omega} \to \widetilde{\Omega}^*$ such that

$$\widetilde{\psi}(x^k y^\lambda \xi^u)(x^l y^\eta \xi^v) := \begin{cases} \psi(x^k y^\lambda \xi^u)(x^l y^\eta \xi^v) & x^k y^\lambda \xi^u, x^l y^\eta \xi^v \in \Omega, \\ 0 & \text{other cases.} \end{cases}$$

As $\widetilde{\Omega} = \Omega \oplus \mathbb{F} x^\pi \xi^\omega$, $\widetilde{\psi}$ is a skew linear mapping of degree l. Then we will show that

$$(3.1) \qquad \widetilde{\psi}([f,g]) = (-1)^{|\widetilde{\psi}||f|} f \cdot \widetilde{\psi}(g) - (-1)^{|\widetilde{\psi}(f)||g|} g \cdot \widetilde{\psi}(f), \ \ \forall f, g \in \widetilde{\Omega}.$$

We shall prove it in two cases:
(i) Consider the case f, $g \in \Omega$. We need only to prove that

$$(3.2) \qquad \widetilde{\psi}([f,g])(x^\pi \xi^\omega) = (-1)^{|\widetilde{\psi}||f|} (f \cdot \widetilde{\psi}(g))(x^\pi \xi^\omega) - (-1)^{|\widetilde{\psi}(f)||g|} (g \cdot \widetilde{\psi}(f))(x^\pi \xi^\omega).$$

By virtue of the definition of $\widetilde{\psi}$, the left-hand side of the equation (3.2) equals 0. Setting $f \in \Omega_i$ and $g \in \Omega_j$, we see that the right-hand side of (3.1) is contained in $(\widetilde{\Omega}^*)_{i+j+l}$. Thereby, the right-hand side of (3.2) coincides with 0 unless $i + j + l = -\tau$, which implies that $-\tau = i + j + l \ge i + j + 5 - \tau \ge 1 - \tau$ for $i, j \ge -2$, a contradiction.
(ii) Consider the case $f = x^\pi \xi^\omega$. By $\widetilde{\Omega} = \oplus_{i=-2}^{\tau} \widetilde{\Omega}_i$ and $\Omega = [\widetilde{\Omega}, \widetilde{\Omega}]$, we have $[x^\pi \xi^\omega, \widetilde{\Omega}] \subset \Omega_\tau \oplus \Omega_{\tau-1} \oplus\ \Omega_{\tau-2}$. Note that $\mathrm{zd}(\widetilde{\psi}([x^\pi \xi^\omega, \widetilde{\Omega}])) \ge l + \tau - 2 \ge (5 - \tau) + \tau - 2 = 3$ and $\widetilde{\psi}([x^\pi \xi^\omega, \widetilde{\Omega}]) \subseteq \widetilde{\Omega}^* = \oplus_{i=-\tau}^{2} (\widetilde{\Omega}^*)_i$. Then we obtain $\widetilde{\psi}([x^\pi \xi^\omega, g]) \in \sum_{i \ge 3} (\widetilde{\Omega}^*)_i = 0$ for $g \in \widetilde{\psi}$. It is easy to see that $x^\pi \xi^\omega \cdot \widetilde{\psi}(g) \subseteq \widetilde{\Omega}^*$ and $g \cdot \widetilde{\psi}(x^\pi \xi^\omega) \subseteq \widetilde{\Omega}^*$. Thus we have $g \cdot \widetilde{\psi}(x^\pi \xi^\omega)(x^k y^\lambda \xi^u) = \alpha \widetilde{\psi}(x^\pi \xi^\omega)([g, x^k y^\lambda \xi^u]) = 0$ for all $x^k y^\lambda \xi^u \in \widetilde{\Omega}$, where $\alpha = \pm 1$ and $x^\pi \xi^\omega \cdot \widetilde{\psi}(g) \in \sum_{i \ge 3} (\widetilde{\Omega}^*)_i = 0$. The proof is complete. □

For $i \in M$, we define a linear mapping $\sigma_i : \widetilde{\Omega} \to \mathbb{F}$ by

$$\sum_{k \le \pi, \lambda \in H, u \in \mathbb{B}(q)} \gamma(k, \lambda, u) x^k y^\lambda \xi^u \to \gamma(\pi - \pi_{i'} e_{i'}, 0, \omega),$$

where $1' = 1$.
For $i \in T$, we define a linear mapping $\sigma_i : \widetilde{\Omega} \to \mathbb{F}$ by

$$\sum_{k \le \pi, \lambda \in H, u \in \mathbb{B}(q)} \gamma(k, \lambda, u) x^k y^\lambda \xi^u \to \gamma(\pi - \pi_1 e_1, 0, \omega \setminus \{i\}).$$

We define a linear mapping $\sigma_\tau : \widetilde{\Omega} \to \mathbb{F}$ by

$$\sum_{k \le \pi, \lambda \in H, u \in \mathbb{B}(q)} \gamma(k, \lambda, u) x^k y^\lambda \xi^u \to \gamma(\pi, 0, \omega).$$

3.2. Proposition. *The following statements hold.*
(1) *The mapping* $\psi_i : \widetilde{\Omega} \longrightarrow \widetilde{\Omega}^*$ *given by* $\psi_i(x^k y^\lambda \xi^u)(x^l y^\eta \xi^v) = \sigma_i(k_i^* x^{k-e_i} y^\lambda \xi^u x^l y^\eta \xi^v)$ *is a skew derivation of degree derivation* $l = p^{s_{i'}+1} - \tau$ *for* $i \in M_1$.
(2) *The mapping* $\psi_i : \widetilde{\Omega} \longrightarrow \widetilde{\Omega}^*$ *given by* $\psi_i(x^k y^\lambda \xi^u)(x^l y^\eta \xi^v) = \sigma_i(\bar{\partial}(x^k y^\lambda \xi^u) \partial_i(x^k y^\lambda \xi^u x^l y^\eta \xi^v))$ *is a skew derivation of degree derivation* $l = 2p^{s_1+1} - \tau$ *for* $i \in T$.
(3) *The mapping* $\psi_1 : \widetilde{\Omega} \longrightarrow \widetilde{\Omega}^*$ *given by* $\psi_1(x^k y^\lambda \xi^u)(x^l y^\eta \xi^v) = \sigma_1(\bar{\partial}(x^k y^\lambda \xi^u) x^k y^\lambda \xi^u x^l y^\eta \xi^v)$

is a skew derivation of degree derivation $l = 2p^{s_1+1} - \tau$.
(4) *The mapping* $\psi_{s+1} : \widetilde{\Omega} \longrightarrow \widetilde{\Omega}^*$ *given by* $\psi_{s+1}(x^k y^\lambda \xi^u)(x^l y^\eta \xi^v) = \sigma_\tau(k_1^* x^{k-e_1} y^\lambda \xi^u x^l y^\eta \xi^v)$ *is a skew derivation of degree derivation* $l = -\tau$.

Proof. Noting that $\widetilde{\Omega}^-$ is generated by $\widetilde{\Omega}_{-1} = \sum_{i=2}^{r-1} \mathbb{F} x_i y^\theta + \sum_{i=r+1}^{s} \mathbb{F} y^\theta \xi_i$ and $p^{s_{i'}+1} > 4$, by Lemma 2.3, it is sufficient to show that for $i \in M \cup T$ and $j \in M_1 \cup T$, the equalities below hold:

$$\psi_i([x_j y^\theta, x^k y^\lambda \xi^u])(x^l y^\eta \xi^v) \\ = (-1)^{|\psi_i||x_j|}(x_j y^\theta \cdot \psi_i(x^k y^\lambda \xi^u))(x^l y^\eta \xi^v) - (-1)^{|\xi^u||\psi_i(x_j y^\theta)|}(x^k y^\lambda \xi^u \cdot \psi_i(x_j y^\theta))(x^l y^\eta \xi^v), \tag{3.3}$$

$$\psi_i([\xi_j y^\theta, x^k y^\lambda \xi^u])(x^l y^\eta \xi^v) \\ = (-1)^{|\psi_i||\xi_j|}(\xi_j y^\theta \cdot \psi_i(x^k y^\lambda \xi^u))(x^l y^\eta \xi^v) - (-1)^{|\xi^u||\psi_i(\xi_j y^\theta)|}(x^k y^\lambda \xi^u \cdot \psi_i(\xi_j y^\theta))(x^l y^\eta \xi^v). \tag{3.4}$$

In case (1)-(3), we only prove (3.3), and (3.4) is treated similarly.

(1) The linear mapping ψ_i is said to be skew if

$$\psi_i(x^k y^\lambda \xi^u)(x^l y^\eta \xi^v) = -(-1)^{|u||v|}\psi_i(x^l y^\eta \xi^v)(x^k y^\lambda \xi^u), \ \forall\ x^k y^\lambda \xi^u, x^l y^\eta \xi^v \in \widetilde{\Omega},\ 2 \le i \le r-1. \tag{3.5}$$

By computing directly, we see that the left-hand side of (3.5) equals $\sigma_i(k_i^* x^{k-e_i} x^l y^{\lambda+\eta} \xi^u \xi^v)$ and the right-hand side of (3.5) coincides with $-\sigma_i(l_i^* x^k x^{l-e_i} y^{\lambda+\eta} \xi^u \xi^v)$. If $k + l - e_i = \pi - \pi_{i'} e_{i'}$, then $k_i + l_i - 1 = \pi_i \equiv -1 \,(\mathrm{mod} p)$; that is, $k_i^* + l_i^* \equiv 0 \,(\mathrm{mod} p)$ for all $i \in M_1$. Hence both sides of (3.5) equal k_i^*. Otherwise, both sides coincide with 0. Therefore, the mapping ψ_i is skew, as desired. Moreover, the linear mapping ψ_i is of degree $l = p^{s_{i'}+1} - \tau$ by a direct computation. For $j \in M_1$, the left-hand side of (3.3) coincides with

$$\psi_i(-k_1^*(1 - \mu_j - \theta)) x_j x^{k-e_1} y^{\lambda+\theta} \xi^u)(x^l y^\eta \xi^v) + \psi_i([j] k_{j'}^* x^{k-e_{j'}} y^{\lambda+\theta} \xi^u)(x^l y^\eta \xi^v),$$

while the right-hand side of (3.3) equals

$$-\psi_i(x^k y^\lambda \xi^u)(-l_1^*(1-\mu_j-\theta) x_j x^{l-e_1} y^{\lambda+\theta} \xi^v) - \psi_i(x^k y^\lambda \xi^u)([j] l_{j'}^* x^{l-e_{j'}} y^{\eta+\theta} \xi^v) + \psi_i(x_j y^\theta)([x^k y^\lambda \xi^u, x^l y^\eta \xi^v]).$$

We distinguish two cases:
Case 1. $i \neq j$.
1.1. $k + l - e_i - e_{j'} = \pi - \pi_{i'} e_{i'}$, $\xi^u \xi^v = \xi^\omega$ and $\lambda + \eta + \theta = 0$.
1.2. $k + l + e_j - e_i - e_1 = \pi - \pi_{i'} e_{i'}$, $\xi^u \xi^v = \xi^\omega$ and $\lambda + \eta + \theta = 0$.
Case 2. $i = j$.
2.1. $k + l - e_i - e_{i'} = \pi - \pi_{i'} e_{i'}$, $\xi^u \xi^v = \xi^\omega$ and $\lambda + \eta + \theta = 0$.
2.2. $k + l - e_1 = \pi - \pi_{i'} e_{i'}$, $\xi^u \xi^v = \xi^\omega$ and $\lambda + \eta + \theta = 0$.

Firstly, we deal with the case 1.1. We see that the left-hand side of (3.3) equals $-[j] k_{j'}^* l_i^*$, while the right-hand side of (3.3) coincides with $-[j] k_i^* l_{j'}^*$. As $k + l - e_i - e_{j'} = \pi - \pi_{i'} e_{i'}$, we get $k_i + l_i - 1 = \pi_i \equiv -1 \,(\mathrm{mod} p)$ and $k_{j'} + l_{j'} - 1 = \pi_{j'} \equiv -1 \,(\mathrm{mod} p)$, i.e., $k_i^* + l_i^* \equiv 0 \,(\mathrm{mod} p)$ and $k_{j'} + l_{j'} \equiv 0 \,(\mathrm{mod} p)$. Thus $-[j] k_{j'}^* l_i^* = -[j] k_i^* l_{j'}^*$ and the equality (3.3) holds.

For the case 1.2, we know that the left-hand side of (3.3) coincides with $k_1^* l_i^* (1 - \mu_j - \theta)$. But the right-hand side of (3.3) equals $l_1^* k_i^* (1 - \mu_j - \theta)$. By the assumptions, we also have $k_i^* + l_i^* \equiv 0 \,(\mathrm{mod} p)$ and $k_1^* + l_1^* \equiv 0 \,(\mathrm{mod} p)$. Consequently, the right-hand side of (3.3) coincides with $k_1^* l_i^* (1 - \mu_j - \theta)$, as desired.

The proof of the case 2.1 is similar to the case 1.2.

For the case 2.2, it is easy to see that the left-hand side of the equation (3.3) coincides with $(1 - \mu_j - \theta) k_1^* l_i^*$. And the right-hand side equals $(1 - \mu_j - \theta) k_i^* l_1^* + (1 - \sum_{i \in M_1} \mu_i l_i - \eta - 2^{-1}|v|) k_1^* - (1 - \sum_{i \in M_1} \mu_i k_i - \lambda - 2^{-1}|u|) l_1^*$. By means of our assumptions, we have $k_1^* + l_1^* \equiv 0 \,(\mathrm{mod} p)$, $k_i + l_i \equiv -1 \,(\mathrm{mod} p)$ and $k_{i'} + l_{i'} = 0$. Hence, the right-hand side of the equation (3.3) equals $l_1^* (1 - \mu_j - \theta)(k_i^* + 1) = (1 - \mu_j - \theta) k_1^* l_i^*$, as desired.

(2) In analogy with (1), it is easily seen that the mapping ψ_i is skew and of degree $2p^{s_1+1}-\tau$. For $j\in M_1$, the left-hand side of (3.3) equals

$$
\begin{aligned}
&\sigma_i(-k_1^*(1-\mu_j-\theta)(1-\sum_{i\in M_1}\mu_ik_i-\mu_j-(\lambda+\theta)-2^{-1}|u|)x_jx^{k-e_1}x^ly^{\lambda+\theta+\eta}\partial_i(\xi^u\xi^v))\\
&+\sigma_i([j]k_{j'}^*(1-\sum_{i\in M_1}\mu_ik_i+\mu_{j'}-(\lambda+\theta)-2^{-1}|u|)x^{k-e_{j'}}x^ly^{\lambda+\eta+\theta}\partial_i(\xi^u\xi^v)),
\end{aligned}
\tag{3.6}
$$

while the right-hand side coincides with

$$
\begin{aligned}
&\sigma_i(l_1^*(1-\mu_j-\theta)(1-\sum_{i\in M_1}\mu_ik_i-\lambda-2^{-1}|u|)x_jx^kx^{l-e_1}y^{\lambda+\theta+\eta}\partial_i(\xi^u\xi^v))\\
&-\sigma_i([j]l_{j'}^*(1-\sum_{i\in M_1}\mu_ik_i-\lambda-2^{-1}|u|)x^kx^{l-e_{j'}}y^{\lambda+\theta+\eta}\partial_i(\xi^u\xi^v))\\
&+\sigma_i(k_1^*(1-\sum_{i\in M_1}\mu_il_i-\eta-2^{-1}|v|)(1-\mu_j-\theta)x_jx^{k-e_1}x^ly^{\lambda+\eta+\theta}\partial_i(\xi^u\xi^v))\\
&-\sigma_i(l_1^*(1-\sum_{i\in M_1}\mu_ik_i-\lambda-2^{-1}|u|)(1-\mu_j-\theta)x_jx^kx^{l-e_1}y^{\lambda+\eta+\theta}\partial_i(\xi^u\xi^v))\\
&+\sigma_i(\sum_{i\in M_1}[i]k_i^*l_{i'}^*(1-\mu_j-\theta)x_jx^{k-e_i}x^{l-e_{i'}}y^{\lambda+\eta+\theta}\partial_i(\xi^u\xi^v))\\
&+\sigma_i(\sum_{i\in T}(-1)^{|\xi^u|}(1-\mu_j-\theta)x_jx^kx^ly^{\lambda+\eta+\theta}\partial_i(D_i(\xi^u)D_{i'}(\xi^v))).
\end{aligned}
\tag{3.7}
$$

We consider the following cases:
Case 1. $k+l+e_j-e_1=\pi-\pi_1e_1$, $\lambda+\eta+\theta=0$ and $\partial_i(\xi^u\xi^v)=\xi^{\omega-\langle i\rangle}$.
Case 2. $k+l-e_{j'}=\pi-\pi_1e_1$, $\lambda+\eta+\theta=0$ and $\partial_i(\xi^u\xi^v)=\xi^{\omega-\langle i\rangle}$.

Now we only prove the Case 2. Then the equation (3.6) equals $[j]k_{j'}^*(1-\sum_{i\in M_1}\mu_ik_i-(\lambda+\theta)+\mu_{j'}-2^{-1}|u|)$, while the equation (3.7) coincides with

$$
[j]l_{j'}^*(1-\sum_{i\in M_1}\mu_ik_i-\lambda-2^{-1}|u|)+[j](1-\mu_j-\theta)k_j^*l_{j'}^*+[j'](1-\mu_j-\theta)k_{j'}^*l_j^*.
\tag{3.8}
$$

As $k+l-e_{j'}=\pi-\pi_1e_1$, $k_{j'}+l_{j'}-1=\pi_{j'}\equiv-1\,(\mathrm{mod}\,p)$. Hence the equation (3.8) coincides with $[j]k_{j'}^*(1-\sum_{i\in M_1}\mu_ik_i-(\lambda+\theta)+\mu_{j'}-2^{-1}|u|)$ and (3.3) is valid.

(3) The linear mapping ψ_1 is clearly skew and of degree $2p^{s_1+1}-\tau$. For the left-hand side of (3.3), we obtain

$$
\begin{aligned}
&\sigma_1(-k_1^*(1-\mu_j-\theta)(1-\sum_{i\in M_1}\mu_ik_i-(\lambda+\theta)-2^{-1}|u|)x_jx^{k-e_1}x^ly^{\lambda+\theta+\eta}\xi^u\xi^v)\\
&+\sigma_1([j]k_{j'}^*(1-\sum_{i\in M_1}\mu_ik_i+\mu_{j'}-(\lambda+\theta)-2^{-1}|u|)x^{k-e_{j'}}x^ly^{\lambda+\theta+\eta}\xi^u\xi^v),
\end{aligned}
\tag{3.9}
$$

and the right-hand side coincides with

$$\sigma_1(l_1^*(1-\mu_j-\theta)(1-\sum_{i\in M_1}\mu_ik_i-\lambda-2^{-1}|u|)x^kx_jx^{l-e_1}y^{\lambda+\theta+\eta}\xi^u\xi^v)$$
$$+\sigma_1(-[j]l_{j'}^*(1-\sum_{i\in M_1}\mu_ik_i-\lambda-2^{-1}|u|)x^kx^{l-e_{j'}}y^{\eta+\theta+\lambda}\xi^u\xi^v)$$
$$+\sigma_1(k_1^*(1-\mu_j-\theta)(1-\sum_{i\in M_1}\mu_il_i-\eta-2^{-1}|v|)x_jx^{k-e_1}x^ly^{\lambda+\eta+\theta}\xi^u\xi^v)$$
$$+\sigma_1(-l_1^*(1-\mu_j-\theta)(1-\sum_{i\in M_1}\mu_ik_i-\lambda-2^{-1}|u|)x_jx^kx^{l-e_1}y^{\lambda+\eta+\theta}\xi^u\xi^v)$$
$$+\sigma_1((1-\mu_j-\theta)\sum_{i\in M_1}[i]k_i^*l_{i'}^*x_jx^{k-e_i}x^{l-e_{i'}}y^{\lambda+\eta+\theta}\xi^u\xi^v)$$
$$+\sigma_1((1-\mu_j-\theta)\sum_{i\in T}(-1)^{|\xi^u|}x_jx^kx^ly^{\lambda+\eta+\theta}D_i(\xi^u)D_{i'}(\xi^v)). \tag{3.10}$$

We treat two cases separately:
(a) $k+l-e_1+e_j=\pi-\pi_1e_1$, $\lambda+\eta+\theta=0$ and $\xi^u\xi^v=\xi^\omega$.
(b) $k+l-e_{j'}=\pi-\pi_1e_1$, $\lambda+\eta+\theta=0$ and $\xi^u\xi^v=\xi^\omega$.

Now we only prove the case (a). By a direct computation, we see that the equation (3.9) coincides with

$$-k_1^*(1-\mu_j-\theta)(1-\sum_{i\in M_1}\mu_ik_i-(\lambda+\theta)-2^{-1}|u|). \tag{3.11}$$

and the equation (3.10) equals

$$k_1^*(1-\mu_j-\theta)(1-\sum_{i\in M_1}\mu_il_i-\eta-2^{-1}|v|). \tag{3.12}$$

Note that $k_i+l_i=\pi_i\equiv-1\,(\mathrm{mod}p)$, $k_j+l_j+1=\pi_j\equiv-1\,(\mathrm{mod}p)$ and $k_1+l_1-1=0$. Then we may assume $k_1=1$ and $l_1=0$, which implies that (3.11) equals $-(1-\mu_j-\theta)(1-\sum_{i\in M_1}\mu_ik_i-(\lambda+\theta)-2^{-1}|u|)$ and the equation (3.12) coincides with $(1-\mu_j-\theta)(1-\sum_{i\in M_1}\mu_il_i-\eta-2^{-1}|v|)$. Since $2n+4-q\equiv0\,(\mathrm{mod}p)$, (3.3) is valid. Suppose $k_1=0$ and $l_1=1$. Obviously, both sides of (3.3) equal 0.

(4) One may easily see that ψ_{s+1} is skew and of degree $-\tau$. For $j\in T$, the left-hand side of (3.4) coincides with

$$\sigma_\tau((2^{-1}-\theta)k_1^*l_1^*x^{k-e_1}x^{l-e_1}y^{\lambda+\eta+\theta}\xi_j\xi^u\xi^v)+\sigma_\tau((-1)^{|u|}k_1^*x^{k-e_1}x^ly^{\lambda+\eta+\theta}\xi^u\partial_j(\xi^v)),$$

while the right-hand side of (3.4) equals

$$\sigma_\tau((2^{-1}-\theta)k_1^*l_1^*x^{k-e_1}x^{l-e_1}y^{\lambda+\eta+\theta}\xi_j\xi^u\xi^v)+\sigma_\tau((-1)^{|u|}k_1^*x^{k-e_1}x^ly^{\lambda+\eta+\theta}\xi^u\partial_j(\xi^v)).$$

Then two cases arise:
(a) $k+l-2e_1=\pi$, $\lambda+\eta+\theta=0$ and $\xi_j\xi^u\xi^v=\xi^\omega$.
(b) $k+l-e_1=\pi$, $\lambda+\eta+\theta=0$ and $\partial_j(\xi^u)\xi^v=\xi^\omega$.

We only deal with the case (a). By a direct computation, both sides coincide with $k_1^*l_1^*(2^{-1}-\theta)$. Then the equation of (3.4) is valid. Similarly, (3.3) is true in case of $j\in M_1$. □

3.3. Lemma. *The following statements hold.*

$$(x_iy^\eta)^{\pi_{i'}}\cdot\psi_i(x_iy^\eta)=\vartheta_i\sigma_\tau,\quad i\in M_1,\,\vartheta_i\in\mathbb{F},\,\eta\in H, \tag{3.13}$$

$$(y^\eta)^{\pi_1}\cdot\psi_1(y^\eta)=\vartheta_1\sigma_\tau,\,\vartheta_1\in\mathbb{F},\,\eta\in H, \tag{3.14}$$

$$(\mathrm{ad}y^\eta\xi_j)^{2\pi_1+1}\cdot\psi_j(y^\eta\xi_j)=\vartheta_j\sigma_\tau,\quad j\in T,\,\vartheta_j\in\mathbb{F},\,\eta\in H, \tag{3.15}$$

where the mapping ψ_i is defined in Proposition 3.2 for $i \in M \cup T$.

Proof. we see that $(\widetilde{\Omega}^*)^{\widetilde{\Omega}} = \mathbb{F}\sigma_\tau$ according to Lemma 2.5. In order to prove (3.13), we only need to show that $(\mathrm{ad}x_i y^\eta)^{p^{s_{i'}+1}}(x^k y^\lambda \xi^u) = 0$ by Lemma 2.5 for all $x^k y^\lambda \xi^u \in \widetilde{\Omega}$. Since $\mathrm{ad}x_i y^\eta = [i]y^\eta D_{i'} + (\eta - \mu_{i'})y^\eta x_i D_1$, we have $(\mathrm{ad}x_i y^\eta)^{p^{s_{i'}+1}} = [i](y^\eta D_{i'})^{p^{s_{i'}+1}} + (\eta - \mu_{i'})^{p^{s_{i'}+1}}(y^\eta x_i D_1)^{p^{s_{i'}+1}}$. By computing step by step, we obtain

$$(y^\eta D_{i'})(x^k y^\lambda \xi^u) = y^\eta D_{i'}(x^k y^\lambda \xi^u) = k_{i'}^* y^\eta x^{k-e_{i'}} y^{\eta+\lambda}\xi^u,$$
$$(y^\eta D_{i'})^2(x^k y^\lambda \xi^u) = y^\eta D_{i'}(k_{i'}^* y^\eta x^{k-e_{i'}} y^{\eta+\lambda}\xi^u) = k_{i'}^*(k_{i'}-1)^* x^{k-2e_{i'}} y^{2\eta+\lambda}\xi^u,$$
$$\cdots\cdots\cdots \qquad \cdots\cdots\cdots \qquad \cdots\cdots\cdots$$
$$(y^\eta D_{i'})^{p^{s_{i'}+1}-1}(x^k y^\lambda \xi^u) = k_{i'}^*(k_{i'}-1)^* \cdots (k_{i'} - \pi_{i'} + 1)^* x^{k-\pi_{i'}e_{i'}} y^{\pi_{i'}\eta+\lambda}\xi^u.$$

Then $(y^\eta D_{i'})^{p^{s_{i'}+1}}(x^k y^\lambda \xi^u) = 0$. Moreover,

$$(y^\eta x_i D_1)(x^k y^\lambda \xi^u) = k_1^* x_i x^{k-e_1} y^{\lambda+\eta}\xi^u,$$
$$(y^\eta x_i D_1)^2(x^k y^\lambda \xi^u) = k_1^*(k_1-1)^* x_i x_i x^{k-2e_1} y^{\lambda+2\eta}\xi^u,$$
$$\cdots\cdots\cdots \qquad \cdots\cdots\cdots \qquad \cdots\cdots\cdots$$
$$(y^\eta x_i D_1)^{p^{s_{i'}+1}-1}(x^k y^\lambda \xi^u) = k_1^*(k_1-1)^* \cdots (k_1-\pi_{i'}+1)^* x_i x_i \cdots x_i x^{k-\pi_{i'}e_1} y^{\pi_{i'}\eta+\lambda}\xi^u.$$

Consequently,

$$(y^\eta x_i D_1)^{p^{s_{i'}+1}}(x^k y^\lambda \xi^u) = k_1^*(k_1-1)^* \cdots (k_1-\pi_{i'})^* x_i x_i \cdots x_i x^{k-(\pi_{i'}+1)e_1} y^{(\pi_{i'}+1)\eta+\lambda}\xi^u = 0.$$

This shows that $(\mathrm{ad}x_i y^\eta)^{p^{s_{i'}+1}} = 0$, as desired.

Similarly, by a direct computation, we get

$$(\mathrm{ad}y^\eta)(x^k y^\lambda \xi^u) = -k_1^*(1-\eta)x^{k-e_1} y^{\lambda+\eta}\xi^u,$$
$$(\mathrm{ad}y^\eta)^2(x^k y^\lambda \xi^u) = -k_1^*(k_1-1)^*(1-\eta)^2 x^{k-2e_1} y^{\eta+2\lambda}\xi^u,$$
$$\cdots\cdots\cdots \qquad \cdots\cdots\cdots \qquad \cdots\cdots\cdots$$
$$(\mathrm{ad}y^\eta)^{p^{s_1+1}-1}(x^k y^\lambda \xi^u) = k_1^*(k_1-1)^* \cdots (k_1 - \pi_1 + 1)^*(1-\eta)^{\pi_1} x^{k-\pi_1 e_1} y^{\eta+\pi_1\lambda}\xi^u.$$

Then $(\mathrm{ad}y^\eta)^{p^{s_1+1}}(x^k y^\lambda \xi^u) = 0$. (3.14) can be proved.

Since $(\mathrm{ad}y^\eta \xi_j)^2(x^k y^\lambda \xi^u) \in \mathbb{F}x^{k-e_1} y^{\lambda+2\eta}\xi^u$, we obtain $(\mathrm{ad}y^\eta \xi_j)^{2p^{s_1+1}} = 0$. Then (3.15) is true according to Lemma 2.5. □

Applying (3.13), (3.14) and (3.15) to $x^\pi \xi^\omega$, we see that

$$\vartheta_i = \kappa \neq 0, \;\; \kappa \in \mathbb{F}, \;\; i \in M_1; \;\; \vartheta_1 = \varrho \neq 0, \;\; \varrho \in \mathbb{F}; \;\; \vartheta_j = \nu \neq 0, \;\; \nu \in \mathbb{F}, \;\; j \in T.$$

3.4. Lemma. *Let $\psi : \widetilde{\Omega} \to \widetilde{\Omega}^*$ be a derivation. Then there are elements $\vartheta_1, \vartheta_2, \cdots, \vartheta_s \in \mathbb{F}$ such that $(\psi - \sum_{i=1}^s \vartheta_i \psi_i)\,|_{\widetilde{\Omega}^-}$ is an inner derivation.*

Proof. Set $L := \widetilde{\Omega}$ and $V := \mathbb{F}x^\pi \xi^\omega$. Put $e_i := x_i y^{\eta^{(i)}}, \; i \in M_1; \;\; e_i := y^{\eta^{(i)}}\xi_i, \; i \in T;$ $e_1 := y^\theta, \; 1' = 1$
and $\chi := (\pi_{1'}, \pi_{2'}, \cdots, \pi_{(r-1)'}, 1, \cdots, 1) \in \mathbb{N}^{s-1}$. Now we show that $W := \{b \mid b \leq \chi\}$ fulfills the conditions of Lemma 2.6.

First we verify the equation $\mathrm{ann}_{U(L^-)^+}(L) = \mathrm{Span}_\mathbb{F}\{e^b \mid b \notin W\}$ holds. Let $e^b \in U(L^-)^+$ and $b \notin W$. Noting that $e_{r+1}, e_{r+2}, \cdots, e_s \in L_{\bar{1}}$, there is an $i \in M$ such that $b_i > \pi_{i'}$. If $i = 1$, then we have $e^b \cdot (x^k y^\lambda \xi^u) = 0$ by $(\mathrm{ad}y^\theta)^{p^{s_1+1}} = 0$. For $i \in M_1$, the proof of Lemma 3.3 and equality (2.4) ensure that $e_i^{p^{s_{i'}+1}} \cdot (x^k y^\lambda \xi^u) = 0$, where $0 \leq k \leq \pi, \lambda \in H, u \in \mathbb{B}(q)$. Thus the inclusion $\mathrm{Span}_\mathbb{F}\{e^b \mid b \notin W\} \subseteq \mathrm{ann}_{U(L^-)^+}(L)$ holds.

Let $v=\sum_{0<b}\beta(b)e^b$ be an element of $\mathrm{ann}_{U(L^-)^+}(L)$. Then $\sum_{b>\chi}\beta(b)e^b\in\mathrm{ann}_{U(L^-)^+}(L)$ follows from the result above. As $v=\sum_{0<b}\beta(b)e^b=\sum_{0<b\leq\chi}\beta(b)e^b+\sum_{b>\chi}\beta(b)e^b\in\mathrm{ann}_{U(L^-)^+}(L)$, $\sum_{0<b\leq\chi}\beta(b)e^b\in\mathrm{ann}_{U(L^-)^+}(L)$. Let $0\neq u:=\sum_{0<b\leq\chi}\beta(b)e^b$. Put $\jmath:=\min\{b_1\mid\beta(b)\neq 0\}$. Then

$$\begin{aligned}
0&=u\cdot x^{\pi+(\jmath-\pi_1)e_1}\xi^\omega\\
&=\sum_{0<b\leq\chi,b_1=\jmath}\beta(b)e^b\cdot x^{\pi+(\jmath-\pi_1)e_1}\xi^\omega\\
&=\sum_{0<b\leq\chi,b_1=\jmath}\beta(b)\alpha(b)(1-\theta)^\jmath\jmath^*(\jmath-1)^*\cdots 1\cdot\\
&\quad\prod_{i=2}^{r-1}\pi_{i'}^*(\pi_{i'}-1)^*\cdots(\pi_{i'}-b_i+1)^*\cdot x^{\pi-\pi_1e_1-b'}e_{r+1}^{b_{r+1}}\cdots e_s^{b_s}y^{\jmath\theta+\eta'}\xi^\omega,
\end{aligned}$$

where $\alpha(b)=\pm1, b'=\{0,b_2e_{2'},\cdots,b_{r-1}e_{(r-1)'}\}$ and $\eta'=b_2\eta_2+b_3\eta_3+\cdots+b_{r-1}\eta_{r-1}$. We see that $x^{\pi-\pi_1e_1-b'}e_{r+1}^{b_{r+1}}\cdots e_s^{b_s}y^{\jmath\theta+\eta'}\xi^\omega\neq 0$, $\jmath^*(\jmath-1)^*\cdots 1\cdot\prod_{i=2}^{r-1}\pi_{i'}^*(\pi_{i'}-1)^*\cdot\cdot\cdot(\pi_{i'}-b_i+1)^*\neq 0$, and $(1-\theta)^\jmath\neq 0$ for $\theta\in H$. Hence $\beta(b)=0$ whenever $b_1=\jmath$, a contradiction. Thus $v\in\mathrm{span}_{\mathbb{F}}\{e^b\mid b\notin W\}$ and the converse inclusion holds. Hence the condition (a) in Lemma 2.6 is satisfied.

Our previous results ensure that $\{e^b\cdot x^\pi\xi^\omega\mid 0\leq b\leq\chi\}$ generates $\widetilde{\Omega}$. Then condition (b) in Lemma 2.6 holds according to $\dim_{\mathbb{F}}\widetilde{\Omega}=2^{|q|}\cdot p^l$, where $l=\sum_{i\in M}(s_i+1)+m$.

Recall that the mapping ψ is defined in Proposition 3.1. Noting that $\Omega=[\widetilde{\Omega},\widetilde{\Omega}]=\langle x^ky^\lambda\xi^u|(k,\lambda,u)\neq(\pi,0,\omega)\rangle$ for $2n+4-q\equiv 0\,(\mathrm{mod}p)$, we obtain $\sigma_\tau([\widetilde{\Omega},\widetilde{\Omega}])=0$. Then by Lemma 2.5, there exist $\vartheta_1,\vartheta_2,\cdots,\vartheta_s\in\mathbb{F}$ such that

$$\begin{aligned}
&(x_iy^{\eta^{(i)}})^{\pi_{i'}}\cdot\psi(x_iy^{\eta^{(i)}})=\vartheta_i^2\sigma_\tau,\ \ i\in M_1,\\
&(y^\theta)^{\pi_1}\cdot\psi(y^\theta)=\vartheta_1^2\sigma_\tau,\\
&(y^{\eta^{(j)}}\xi_j)^{2\pi_1+1}\cdot\psi(y^{\eta^{(j)}}\xi_j)=\vartheta_j^2\sigma_\tau,\ \ j\in T.
\end{aligned}$$

Put $\phi:=\psi-\sum_{j=1}^s\vartheta_j\psi_j$. Then by computing and Lemma 3.3, we have $(e_i)^{2\pi_1+1}\cdot\phi(e_i)=0$ for $i\in T$ and $(e_i)^{\pi_{i'}}\cdot\phi(e_i)=0$ for $i\in M_1$. The assertion now follows by applying (2) and (3) of Proposition 2.6. □

3.5. Lemma. *Let $d\geq 3$. Then $B(\widetilde{\Omega})_d=\widetilde{\Omega}_d$ for $d\not\equiv 0,-2\,(\mathrm{mod}p)$.*

Proof. Put $B:=B(\widetilde{\Omega})=[\widetilde{\Omega}^+,\widetilde{\Omega}^+]$, where $\widetilde{\Omega}^+=\sum_{i=1}^\tau(\widetilde{\Omega})_i$ is the subalgebra of $\tilde{\Omega}$. The inclusion "$\subseteq$" of our assertion is obvious. The converse inclusion will be proved by considering the following cases. Let $x^ky^\lambda\xi^u\in\widetilde{\Omega}_d$.

(i) $k_1=1$.

(1) $i\notin\{u\}$ and $\{u\}\neq\{\omega\}$. We have $[x_1\xi_i,x^{k-e_1}y^\lambda\xi_i\xi^u]=-x^ky^\lambda\xi^u$. Since $[x_1\xi_i,x^{k-e_1}y^\lambda\xi_i\xi^u]\subseteq[\widetilde{\Omega}^+,\widetilde{\Omega}^+]_d=B_d$, $x^ky^\lambda\xi^u\in B_d$.

(2) $\{u\}\cup\{v\}\cup\{i\}=\{\omega\}$. suppose $|\omega|\geq 4$. If $1-\lambda-2^{-1}|v|\not\equiv 0\,(\mathrm{mod}p)$, then by $[x_1^2x^{k-e_1}\xi_i\xi^u,y^\lambda\xi^v]=2(1-\lambda-2^{-1}|v|)x^ky^\lambda\xi^\omega$, we get $x^ky^\lambda\xi^\omega\in B_d$. If $1-\lambda-2^{-1}|v|\equiv 0\,(\mathrm{mod}p)$, then $1-\lambda-2^{-1}(|v|+1)=-2^{-1}\not\equiv 0\,(\mathrm{mod}p)$. Since $[x_1^2x^{k-e_1}\xi_i\xi^{u-\langle j\rangle},y^\lambda\xi_j\xi^v]=2\alpha(1-\lambda-2^{-1}(|v|+1))x^ky^\lambda\xi^\omega$ with $\alpha=\pm1$, $x^ky^\lambda\xi^\omega\in B_d$, as desired.

Let $|\omega|=3$. Then by $[x_1^2x^{k-e_1}y^\lambda,\xi^\omega]=-x^ky^\lambda\xi^\omega$, we obtain $x^ky^\lambda\xi^\omega\in B_d$.

Let $|\omega|=2$. If $1-\sum_{j\in M_1}\mu_jk_j-\lambda\not\equiv 0\,(\mathrm{mod}p)$, then by means of $[x^ky^\lambda,x_1\xi^\omega]=-(1-\sum_{j\in M_1}\mu_jk_j-\lambda)x^ky^\lambda\xi^\omega$, one gets $x^ky^\lambda\xi^\omega\in B_d$. If $1-\sum_{j\in M_1}\mu_jk_j-\lambda\equiv 0\,(\mathrm{mod}p)$, then $1-(1-\sum_{j\in M_1}\mu_jk_j-\lambda)=1\not\equiv 0\,(\mathrm{mod}p)$. As $[x^ky^\lambda\xi_1,x_1\xi_2]=(1-(1-\sum_{j\in M_1}\mu_jk_j-\lambda))x^ky^\lambda\xi_1\xi_2$, $x^ky^\lambda\xi_1\xi_2\in B_d$ is valid.

(ii) $k_1 \geq 2$.
(1) $\varepsilon_0(k_1-1) \neq p-1$. If $\{u\} \neq \{\omega\}$ and $i \notin \{u\}$, then $[x_1\xi_i, x^{k-e_1}y^\lambda\xi_i\xi^u] = -x^ky^\lambda\xi^u$ implies that $x^ky^\lambda\xi^u \in B_d$.

Let $\{u\} = \{\omega\}$ and $|\omega| \geq 2$. According to

$$(3.16)\ [x_1^2, x^{k-e_1}y^\lambda\xi^\omega] = \big[2(1-\sum_{j\in M_1}\mu_jk_j - \lambda - 2^{-1}|\omega|) - (k_1-1)^*\big]x^ky^\lambda\xi^\omega,$$

we obtain $x^ky^\lambda\xi^\omega \in B_d$ whenever $2(1-\sum_{j\in M_1}\mu_jk_j-\lambda-2^{-1}|\omega|)-(k_1-1)^* \not\equiv 0\,(\mathrm{mod} p)$. If $2(1-\sum_{j\in M_1}\mu_jk_j-\lambda-2^{-1}|\omega|)-(k_1-1)^* \equiv 0\,(\mathrm{mod} p)$, then we consider the following equation

$$[x_1^2\xi_{i_1}, x^{k-e_1}y^\lambda\xi^{\omega-\langle i_1\rangle}] = 2^{-1}((k_1-1)^*+2)x^ky^\lambda\xi^\omega.$$

If $(k_1-1)^*+2 \not\equiv 0\,(\mathrm{mod} p)$, then we have $x^ky^\lambda\xi^\omega \in B_d$.

If $(k_1-1)^*+2 \equiv 0\,(\mathrm{mod} p)$, then $(k_1-1)^* = p-2$, which means that two cases arise:
(a) $\varepsilon_0(k_1-1) = 0$, i.e., $k_1^* = 1 = \varepsilon_0(k_1)$,
(b) $\varepsilon_0(k_1-1) = p-2$, i.e., $k_1^* = p-1 = \varepsilon_0(k_1)$.

Consider the case (a). By$[x_1\xi_{i_1}, x^ky^\lambda\xi^{\omega-\langle i_1\rangle}] = -x^ky^\lambda\xi^\omega$, we obtain $x^ky^\lambda\xi^\omega \in B_d$.

Consider the case (b). Let $|\omega| \geq 3$.

If $k_1 \neq \pi_1$, then by $[x^{k+e_1}y^\lambda\xi^{u_1}, \xi^{u_2}] = -2^{-1}(k_1+1)^*x^ky^\lambda\xi^\omega$ and $(k_1+1)^* \not\equiv 0\,(\mathrm{mod} p)$, we have $x^ky^\lambda\xi^\omega \in B_d$, where $|u_2| = 3$ and $\{u_1\}\cup\{u_2\} = \{\omega\}$.

If $k_1 = \pi_1$, then there exists an i such that $k_i \neq \pi_i$. Otherwise, we obtain

$$\begin{aligned} d &= \sum_{i\in M_1}\pi_i + 2\pi_1 - 2 + |\omega| \\ &= \sum_{i\in M_1}(p^{t_i+1}-1) + 2(p^{t_1+1}-1) - 2 + q \\ &\equiv -(2n+4-q) \equiv 0\,(\mathrm{mod} p), \end{aligned}$$

contradicting $d \not\equiv 0, -2\,(\mathrm{mod} p)$. Hence we have

$$[x^{k+e_i}y^\lambda, x_{i'}\xi^\omega] = \big(1-\mu_{i'}k_{i'}-2^{-1}|\omega|\big)k_1^*x^{k+e_i+e_{i'}-e_1}y^\lambda\xi^\omega + [i](k_i+1)^*x^ky^\lambda\xi^\omega.$$

Set $k+e_i+e_{i'}-e_1 = l$. If $(1-\mu_{i'}k_{i'}-2^{-1}|\omega|) \not\equiv 0\,(\mathrm{mod} p)$, then $\varepsilon_0(l_1) = p-2$ yields $2(1-\sum_{j\in M_1}\mu_jk_j-\lambda-2^{-1}|\omega|)-(k_1-1)^* \not\equiv 0\,(\mathrm{mod} p)$. According to (3.16), we get $x^ly^\lambda\xi^\omega \in B_d$. Otherwise, the first term on the right-hand side of the equation above coincides with 0. Since $(k_i+1)^* \not\equiv 0\,(\mathrm{mod} p)$, $x^ky^\lambda\xi^\omega \in B_d$ is valid.

Let $|\omega| = 2$. If $\varepsilon_0(k_i) = p-1$ for all i, then $d = \sum_{i\in M_1}k_i + 2k_1 - 2 + |\omega| \equiv -(2n+4-q) \equiv 0\,(\mathrm{mod} p)$, contradicting $d \not\equiv 0, -2\,(\mathrm{mod} p)$. Consequently, we have

$$[x^{k+e_i}y^\lambda, x_{i'}\xi^\omega] = -k_1^*\mu_{i'}k_{i'}x^{k+e_i+e_{i'}-e_1}y^\lambda\xi^\omega + [i](k_i+1)^*x^ky^\lambda\xi^\omega.$$

Put $k+e_i+e_{i'}-e_1 = l$. Then $x^ky^\lambda\xi^\omega \in B_d$, which is completely analogous to the proof above.
(2) $\varepsilon_0(k_1-1) = p-1$. Let $\{u\} = \{\omega\}$. If $1-\sum_{j\in M_1}\mu_jk_j-\lambda-2^{-1}(|\omega|-1) \not\equiv 0\,(\mathrm{mod} p)$, then by $[x_1\xi_{i_1}, x^ky^\lambda\xi^{\omega-\langle i_1\rangle}] = (1-\sum_{j\in M_1}\mu_jk_j-\lambda-2^{-1}(|\omega|-1))x^ky^\lambda\xi^\omega$, we get $x^ky^\lambda\xi^\omega \in B_d$. If $1-\sum_{j\in M_1}\mu_jk_j-\lambda-2^{-1}(|\omega|-1) \equiv 0\,(\mathrm{mod} p)$, then $1-\sum_{j\in M_1}\mu_jk_j-\lambda-2^{-1}(|\omega|-2) \not\equiv 0\,(\mathrm{mod} p)$. Since $[x_1\xi_{i_1}\xi_{i_2}, x^ky^\lambda\xi^{\omega-\langle i_1\rangle-\langle i_2\rangle}] = (1-\sum_{j\in M_1}\mu_jk_j-\lambda-2^{-1}(|\omega|-2))x^ky^\lambda\xi^\omega$, $x^ky^\lambda\xi^\omega \in B_d$, as desired.

Let $\{u\} \neq \{\omega\}$. If there exists an $i \in M_1$ such that $\varepsilon_0(k_i) \neq 0$, then by $[x_i^2\xi_i, x^{k-2e_i}y^\lambda\xi_i\xi^u] = -x^ky^\lambda\xi^u$, we obtain $x^ky^\lambda\xi^u \in B_d$. Let $\varepsilon_0(k_i) = 0$ for all $i \in M_1$. If $\sum_{i\in M_1}k_i + 2k_1 - 2 \geq 1$, it is easily seen that $x^ky^\lambda\xi^u \in B_d$ by $[x_1y^\lambda\xi^u, x^k] = x^ky^\lambda\xi^u$. Let $\sum_{i\in M_1}k_i + 2k_1 - 2 \leq 0$. The assumption $d \geq 3$ in this proposition implies that $|u| \geq 3$. Then by $[x_1\xi_{i_1}, x^ky^\lambda\xi^{u-\langle i_1\rangle}] = (1-\lambda-2^{-1}|u|+2^{-1})x^ky^\lambda\xi^u$, we obtain $x^ky^\lambda\xi^u \in B_d$ if

$1-\lambda-2^{-1}|u|+2^{-1}\not\equiv 0\,(\mathrm{mod}p)$. Since $1-\lambda-2^{-1}|u|+2^{-1}\equiv 0\,(\mathrm{mod}p)$ implies $1-\lambda-2^{-1}|u|+1\not\equiv 0\,(\mathrm{mod}p)$, we obtain $x^ky^\lambda\xi^u\in B_d$ by $[x_1\xi_{i_1}\xi_{i_2}, x^ky^\lambda\xi^{u-\langle i_1\rangle-\langle i_2\rangle}]=(1-\lambda-2^{-1}|u|+1)x^ky^\lambda\xi^u$.

(iii) $k_1=0$.

For $|u|\geq 1$, if $1-\sum_{j\in M_1}\mu_jk_j-\lambda-2^{-1}(|u|-1)\not\equiv 0\,(\mathrm{mod}p)$, then by means of $[x_1\xi_{i_1}, x^ky^\lambda\xi^{u-\langle i_1\rangle}]=(1-\sum_{j\in M_1}\mu_jk_j-\lambda-2^{-1}(|u|-1))x^ky^\lambda\xi^u$, we see that $x^ky^\lambda\xi^u\in B_d$.

For $|u|\geq 2$, if $1-\sum_{j\in M_1}\mu_jk_j-\lambda-2^{-1}(|u|-1)\equiv 0\,(\mathrm{mod}p)$, then $1-\sum_{j\in M_1}\mu_jk_j-\lambda-2^{-1}(|u|-2)=2^{-1}\not\equiv 0\,(\mathrm{mod}p)$. According to $[x_1\xi_{i_1}\xi_{i_2}, x^ky^\lambda\xi^{u-\langle i_1\rangle-\langle i_2\rangle}]=(1-\sum_{j\in M_1}\mu_jk_j-\lambda-2^{-1}(|u|-2))x^ky^\lambda\xi^u$, we have $x^ky^\lambda\xi^u\in B_d$.

For $|u|=0$, the assumption $d\geq 3$ in this proposition implies that $\sum_{i\in M_1}k_i+2k_1-2\geq 3$. If $x_ix^{k-e_i}=0$ for all i, then $\varepsilon_0(k_i)=0$. Hence $d=\sum_{i\in M_1}k_i+2k_1-2=\sum_{i\in M_1}k_i-2\equiv -2\,(\mathrm{mod}p)$, contradicting $d\not\equiv 0,-2\,(\mathrm{mod}p)$. Thus there exists an $i\in M_1$ such that $x_ix^{k-e_i}\neq 0$, which implies $x_i^2x^{k-2e_i}\neq 0$. Then we get $x^ky^\lambda\in B_d$ by virtue of $[x_i^3, x^{k-2e_i+e_{i'}}y^\lambda]=3\alpha(k_{i'}+1)^*x^ky^\lambda$, where $\alpha=\pm 1$. □

3.6. Proposition. *The algebra Ω dose not possess a nondegenerate associative for $2n+4-q\equiv 0\,(\mathrm{mod}p)$.*

Proof. We know that $\Omega=\bigoplus_{i=-2}^{\tau}\Omega_i$, where $\Omega_\tau=\mathrm{span}_{\mathbb{F}}\{x^\pi y^\eta\xi^\omega\mid \eta\in H\backslash\{0\}\}$. Clearly $\dim\Omega_\tau=p^m-1$ and $\dim\Omega_{-2}=p^m$. Thus $\dim\Omega_\tau\neq\dim\Omega_{-2}$. Then our assertion is true by Proposition 2.1 in [10]. □

3.7. Theorem. *The second cohomology group $H^2(\Omega,\mathbb{F})$ is $(s+1)$-dimensional.*

Proof. It was proved in [9] that $H^2(L,\mathbb{F})$ is isomorphic to the vector space of skew outer derivations from L into L^* if the modular Lie superalgebra L is simple and does not admit any nondegenerate associative form. We see that Ω is simple (see [13]) and has no nondegenerate associative form according to Proposition 3.6. We propose to show that the vector space V of skew derivations from Ω to Ω^* decomposes as

$$V=\oplus_{i=1}^{s+1}\mathbb{F}\psi_i\oplus\mathrm{Inn}_{\mathbb{F}}(\Omega,\Omega^*),$$

where ψ_i defined in Proposition 3.2 is regarded as a skew derivation from Ω to Ω^* for $i\in M\cup T$.

Let $\psi\in V$ of degree l. Then $-2\tau\leq l\leq 4$. For $-(\tau-1)\leq l\leq 4$, by corresponding ψ to the root space decomposition, we obtain $\psi=0$ or $l\equiv 0\,(\mathrm{mod}p)$. As $\tau\equiv 0\,(\mathrm{mod}p)$, we have $\psi=0$ for $l=-\tau+1,-\tau+2,-\tau+3,-\tau+4$. Let $5-\tau\leq l\leq 4$. According to Proposition 3.1, ψ can be extended to a skew derivation $\widetilde{\psi}:\widetilde{\Omega}\to\widetilde{\Omega}^*$. By Lemma 3.4, it follows that there are $\vartheta_1,\vartheta_2,\cdots,\vartheta_s\in\mathbb{F}$ and $f\in\widetilde{\Omega}^*$ such that

$$\widetilde{\psi}(z)=\sum_{i=1}^{s}\vartheta_i\psi_i(z)+(-1)^{|z||f|}z\cdot f,\quad\forall\ z\in\widetilde{\Omega}^-.$$

Put $g:=f\mid_\Omega$. Then $\psi(z)=\sum_{i=1}^{s}\vartheta_i\psi_i(z)+(-1)^{|z||g|}z\cdot g$ for all $z\in\Omega^-$. Hence, $\psi-\Sigma_{i=1}^{s}\vartheta_i\psi_i\in\mathrm{Inn}(\Omega,\Omega^*)$ by virtue of Lemma 2.7.

For $-2\tau\leq l\leq -\tau$. According to the proof above, we see that $\psi=0$ or $l\equiv 0\,(\mathrm{mod}p)$. Then $\psi=0$ for $l=-\tau-1\not\equiv 0\,(\mathrm{mod}p)$. Thus $-2\tau\leq l\leq -\tau-2$ and $l=-\tau$. We first consider the case $-2\tau\leq l\leq -\tau-2$. Note that $-(\tau-1+l)\geq 3$ and $-(\tau-1+l)\not\equiv 0,-2\,(\mathrm{mod}p)$. Then Lemmas 2.8 and 3.5 ensure that $\psi=0$. For the case $l=-\tau$, we define a bilinear symmetric form $\zeta:\Omega\times\Omega\to\mathbb{F}$ given by

$$\zeta(x^ky^\lambda\xi^u, x^ly^\eta\xi^v)=\sigma_\tau(x^kx^ly^{\lambda+\eta}\xi^u\xi^v).$$

It is easily seen that $\mathrm{rad}(\zeta) = \{x \in \Omega \mid \zeta(x, y) = 0,\ \forall\, y \in \Omega\} = \mathbb{F}1$. Let $\varpi : \Omega \to \widetilde{V}$, $\widetilde{V} := \Omega/\mathbb{F}1$, be the canonical projection. We denote by ρ the bilinear form on $\widetilde{V}$ which is induced by ζ. One may easily verify that the results of Theorems 3.3 and 3.7 in paper [3] are also true for Lie superalgebras. It follows that there is a unique skew p-module homomorphism $D : \widetilde{V} \to \widetilde{V}$ of degree -2 such that

$$\psi(x)(y) = \rho(D(\varpi(x)), \varpi(y)), \quad \forall\, x, y \in \Omega,$$

where

$$P := \quad \Omega^- \oplus \mathrm{span}_{\mathbb{F}}\{x_i x_j \mid 2 \le i, j \le r-1\} \oplus \mathrm{span}_{\mathbb{F}}\{x_i \xi_j \mid 2 \le i \le r-1, r+1 \le j \le s\}$$
$$\oplus \mathrm{span}_{\mathbb{F}}\{\xi_i \xi_j \mid r+1 \le i < j \le s\}.$$

Clearly, the mapping D is uniquely determined by $D(\varpi(x^{\pi-e_2}\xi^{\omega}))$. A direct computation entails the existence of $\beta \in \mathbb{F}$ with $D(\varpi(x^{\pi-e_2}\xi^{\omega})) = \beta\varpi(x^{\pi-e_1-e_2}\xi^{\omega})$ by the degree of D. As a result, $D(v) = 2^{-1}\beta \cdot 1 \cdot v$ for $v \in \widetilde{V}$, and $\psi = \beta\psi_{s+1}$. Consequently, the dimension of the vector space of skew outer derivations of Ω is $s+1$ and our assertion is true. □

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