Some properties of $q-$ close-to-convex functions

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Abstract
Quantum calculus had been used first time by M.E.H.Ismail, E.Merkes and D.Steyr in the theory of univalent functions [5]. In this present paper we examine the subclass of univalent functions which is defined by quantum calculus.

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1. Introduction
Let $Ω$ be the family of functions $φ(z)$ regular in the open unit disc $D = \{z : |z| < 1\}$ and satisfy the conditions $φ(0) = 0$, $|φ(z)| < 1$ for all $z \in D$. Denote by $P(q)$ the family of functions of the form $p(z) = 1 + p_1z + p_2z^2 + \cdots$ which are regular in the open unit disc $D$ and satisfying

\begin{equation}
|p(z) - \frac{1}{1-q}| < \frac{1}{1-q}, \quad z \in D
\end{equation}

where $q \in (0, 1)$ is a fixed real number. Let $A$ be the family of functions $f(z)$ which are regular in the open unit disc $D$ and satisfying the conditions $f(0) = 0$, $f'(0) = 1$ for every $z \in D$. In other words; each $f$ in $A$ has the power series representation $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Let $f_1(z)$ and $f_2(z)$ be an elements of $A$, if there exists a function $φ(z) \in Ω$, such that $f_1(z) = f_2(φ(z))$ for all $z \in D$, then we say that $f_1(z)$ is subordinate to $f_2(z)$ and we write $f_1(z) \prec f_2(z)$, thus $f_1(z) \prec f_2(z)$ if and

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only if \( f_1(0) = f_2(0) \) and \( f_1(\mathbb{D}) \subset f_2(\mathbb{D}) \) implies \( f_1(\mathbb{D}_r) \subset f_2(\mathbb{D}_r) \), where \( \mathbb{D}_r \) defined as 
\[
\mathbb{D}_r = \{ z : |z| < r, 0 < r < 1 \} \) (Subordination principle[4]).

In this paragraph we will give the concept of the \( q- \) calculus. Let \( q \in (0, 1) \) be a fixed number. A subset of \( \mathbb{B} \) of \( \mathbb{C} \) is called \( q- \) geometric if \( qz \in \mathbb{B} \) whenever \( z \in \mathbb{B} \), if a subset \( \mathbb{B} \) of \( \mathbb{C} \) is a \( q- \) geometric set, then it contains all geometric sequences \( \{ q^n \} \), \( zq \in \mathbb{B} \). Let \( f \) be a function (real or complex valued) defined on \( q- \) geometric set \( \mathbb{B} \), \( |q| \neq 1 \), the \( q- \) difference operator which was introduced by Jackson F.H. and E.Heine or Euler([1],[2],[3],[7]) defined by

\[
(1.2) \quad D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \quad z \in \mathbb{B}\setminus\{0\}
\]

The \( q- \) difference operator (1.2) sometimes called Jackson \( q- \) difference operator. If \( 0 \in \mathbb{B} \), the \( q- \) derivative at zero by \( |q| < 1 \)

\[
(1.3) \quad D_q f(0) = \lim_{n \to \infty} \frac{f(zq^n) - f(0)}{zq^n}, \quad z \in \mathbb{B}\setminus\{0\}
\]

provided the limit exists, and does not depend on \( z \) in addition \( q- \) derivative at zero defined by \( |q| < 1 \)

\[
(1.4) \quad D_q f(0) = D_{q^{-1}} f(0)
\]

Under the hypothesis of the definition of \( q- \) difference operator, then we have the following rules:

1. For a function \( f(z) = z^n \) we observe that \( D_q f(z) = D_q z^n = \frac{1-q^n}{1-q} z^{n-1} \), therefore we have \( f(z) = z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + \cdots \Rightarrow \)

\[
D_q f(z) = 1 + \sum_{n=2}^{\infty} a_n \frac{1-q^n}{1-q} z^{n-1}
\]

2. Let \( f(z) \) and \( g(z) \) be defined on a \( q- \) geometric set \( \mathbb{B} \subset \mathbb{C} \) such that \( q \) derivatives of \( f \) and \( g \) exist for all \( z \in \mathbb{B} \), then

(i) \( D_q (af(z) \pm bg(z)) = aD_q f(z) \pm bD_q g(z) \) where \( a \) and \( b \) are real or complex constants

(ii) \( D_q (f(z)g(z)) = g(z)D_q f(z) + f(qz)D_q g(z) \)

(iii) \( D_q \left( \frac{g(z)}{h(z)} \right) = \frac{g(z)D_q h(z) - h(z)D_q g(z)}{h(z)h(qz)} = \frac{g(qz)D_q h(z) - h(z)D_q g(z)}{h(z)h(qz)} \)

where \( h(z)h(qz) \neq 0 \).

(iv) As a right inverse, Jackson([1],[2],[3],[7]) introduced the \( q- \) integral

\[
\int_0^z f(t) d_q t = z(1-q) \sum_{n=0}^{\infty} q^n f(q^n z)
\]

provided that the series converges. The following theorem is an analogue of the fundamental theorem of calculus.

1.1. **Theorem** ([7]). Let \( f \) be a \( q- \) regular at zero, defined on \( q- \) geometric set \( \mathbb{B} \) containing zero. Define

\[
F(z) = \int_{c}^{z} f(t) d_q t, \quad (z \in \mathbb{B})
\]
where $c$ is a fixed point in $B$, then $F$ is a regular at zero. Furthermore $D_q F(z)$ exists for every $z \in B$ and

$$D_q F(z) = f(z)$$

for all $z \in B$.

Conversely, If $a$ and $b$ are two points in $B$, then

$$\int_a^b D_q f(z) dz = f(b) - f(a)$$

(3) The $q-$ differential is defined as

$$d_q f(z) = f(z) - f(qz)$$

therefore

$$D_q f(z) = \frac{d_q f(z)}{d_q z} = \frac{f(z) - f(qz)}{(1-q)z} \Rightarrow d_q f(z) = \frac{f(z) - f(qz)}{(1-q)z} d_q z.$$

(4) The partial $q-$ derivative of a multivariable real continuous function $f(x_1, x_2, x_3, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$ to a variable $x_i$ defined by

$$D_{q,x_i} F(\vec{x}) = \frac{f(\vec{x}) - \varepsilon_{q,x_i} f(\vec{x})}{(1-q)x_i}, x_i \neq 0, q \in (0, 1)$$

$$[D_{q,x_i} F(\vec{x})]_{x_i=0} = \lim_{x_i \to 0} D_{q,x_i} F(\vec{x})$$

where $\varepsilon_{q,x_i} f(\vec{x}) = f(x_1, x_2, x_3, \ldots, x_{i-1}, qx_i, x_{i+1}, \ldots, x_n)$ and we use $D^k_{q,x}$ instead of operator $\frac{\partial^k}{\partial q^k}$ for some simplification.

1.2. Lemma. ([6] JACK'S LEMMA) Let $\phi(z)$ be analytic in $D$ with $\phi(0) = 0$. If the maximum value of the $|\phi(z)|$ on the circle $|z| = r$, $(0 < r < 1)$ is attained at $z = z_0$, then we have

$$z_0 \phi'(z_0) = m\phi(z_0), m \geq 1$$

Making use of the $q-$ derivative $D_q f(z)$, we introduce the following classes.

$$S_q^* = \{f(z) \in A | z \frac{D_q f(z)}{f(z)} = p(z), p(z) \in P(q)\}$$

(The class of $q-$ starlike functions [5].)

$$C_q = \{f(z) \in A | \frac{D_q (zD_q f(z))}{D_q f(z)} = p(z), p(z) \in P(q)\}$$

(The class of $q-$ convex functions)

$$K_q = \{g(z) \in A | \frac{D_q g(z)}{D_q f(z)} = p(z), p(z) \in P(q), f(z) \in C_q\}$$

(The class of $q-$ close-to-convex functions.)

In the present paper we will investigate the class of $K_q$. 

2. Main Results

2.1. Theorem ([8]). \( p(z) \in P(q) \) if and only if

\[
p(z) < \frac{1 + z}{1 - qz}
\]

Proof. Let \( p(z) \) be an element of \( P(q) \) then we have

\[
|p(z) - \frac{1}{1 - q}| < \frac{1}{1 - q} \Rightarrow |p(z) - m| < m
\]

where

\[
\frac{1}{1 - q} = m \iff 1 - q = \frac{1}{m} \Rightarrow 1 - \frac{1}{m} = q.
\]

Therefore the function

\[
\psi(z) = \frac{1}{m} p(z) - 1
\]

has modulus at most 1 in the open unit disc \( \mathbb{D} \) and so

\[
\phi(z) = \frac{\psi(z) - \psi(0)}{1 - \psi(0)\psi(z)} = \frac{\left(\frac{1}{m} p(z) - 1\right) - \left(\frac{1}{m} - 1\right)}{1 - \left(\frac{1}{m} - 1\right)\left(\frac{1}{m} p(z) - 1\right)}
\]

satisfies the conditions of Schwarz Lemma, this shows that

\[
p(z) = \frac{1 + \phi(z)}{1 - \left(1 - \frac{1}{m}\right)\phi(z)} \Rightarrow p(z) < \frac{1 + z}{1 - qz}.
\]

Conversely; Suppose that the function \( p(z) \) analytic in \( \mathbb{D} \) and satisfies the condition \( p(0) = 1 \) and

\[
p(z) < \frac{1 + z}{1 - qz}
\]

then we have

\[
p(z) < \frac{1 + z}{1 - qz} \Rightarrow p(z) = \frac{1 + \phi(z)}{1 - \left(1 - \frac{1}{m}\right)\phi(z)} \Rightarrow p(z) - m = \frac{1 - m + \phi(z)}{1 + \frac{1 - m}{m} \phi(z)}.
\]

On the other hand the function \( \left(\frac{1 - m + \phi(z)}{1 + \frac{1 - m}{m} \phi(z)}\right) \) maps the unit circle onto itself, then we have

\[
|p(z) - m| = \left| m \frac{1 - m + \phi(z)}{1 + \frac{1 - m}{m} \phi(z)} \right| < m.
\]

This shows that \( p(z) \in P(q) \).

\[\square\]

2.2. Lemma ([8]). Let \( f(z) \) be a function (real or complex valued) defined on \( q \)-geometric set \( \mathbb{B} \) with \( |q| \neq 1 \), then

(2.1) \( D_q(\log f(z)) = \frac{D_q f(z)}{f(z)} \)

Proof. Using the definition of \( q \)-difference operator, then we have

\[
D_q(\log f(z)) = \frac{\log f(z) - \log f(qz)}{(1 - q)z} = \log \left(1 + \frac{1}{h} D_q f(z) \right)^{\frac{1}{h}}
\]

Taking limit for \( h \to 0 \) we obtain (2.1)

\[\square\]

2.3. Lemma. (\( q \)-Jack’s Lemma [8]) Let \( \phi(z) \) be analytic in \( \mathbb{D} \) with \( \phi(0) = 0 \). Then if \( |\phi(z)| \) attains its maximum value on the circle \( |z| = r \) at a point \( z_0 \in \mathbb{D} \), then we have

\[
z_0 D_q \phi(z_0) = m \phi(z_0),
\]

where \( m \geq 1 \) is a real number.
2.5. Theorem

We need to show that

But this is a contradict with (2.4). Therefore

Using Jack's Lemma in (2.5)

Therefore we have

$\frac{zq}{1 - qz}$

2.4. Theorem ([8]). Let $f(z)$ be an element of $C_q$, then

(2.2) \[ \frac{z D_q f(z)}{f(z)} < \frac{1}{1 - qz} \]

Proof. We define the function $\phi(z)$ by

(2.3) \[ \frac{z D_q f(z)}{f(z)} = \frac{1}{1 - q\phi(z)} \]

Since $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$, $z D_q f(z) = z + a_2 \frac{1 - q^2}{1 - q} z^2 + a_3 \frac{1 - q^3}{1 - q} z^3 + \cdots$, then $\phi(z)$ is well defined and analytic at the same time

\[ \frac{z D_q f(z)}{f(z)} \bigg|_{z=0} = 1 = \frac{1}{1 - q\phi(z)} \Rightarrow \phi(0) = 0 \]

We need to show that $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. Assume to the contrary, then there exists a $z_0 \in \mathbb{D}$ such that $|\phi(z_0)| = 1$. The definition of the class $C_q$ and subordination principle, then we write

(2.4) \[ A(r) = \left\{ f(z) : \left| (1 + qz) \frac{D_q f(z)}{f(z)} \right| = \frac{1 + qz}{1 - qz} \leq \frac{(1 + q)r}{1 - q^2r^2}, f(z) \in C_q \right\} \]

On the other hand, using the definition $q-$ derivative, theorem 2.1 and the relation (2.3) and after the straightforward calculations we get

(2.5) \[ \frac{(1 + qz)}{D_q f(z)} = q \left( \frac{1}{1 - q\phi(z)} \right) + \frac{logq^{-1}}{1 - q} z D_q \phi(z) + \left( 1 - \frac{logq^{-1}}{1 - q} \right) \]

Using $q-$Jack's Lemma in (2.5)

(1 + qz_0) \frac{D_q f(z_0)}{f(z_0)} = q \left( \frac{1}{1 - q\phi(z_0)} \right) + \frac{logq^{-1}}{1 - q} m\phi(z_0) + \left( 1 - \frac{logq^{-1}}{1 - q} \right) \notin A(r).

But this is a contradict with (2.4). Therefore $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. \(\square\)

2.5. Theorem ([8]). Let $f(z)$ be an element of $C_q$, then

(2.6) \[ \frac{r}{(1 + qr)^{\frac{1 + q}{1 - q}}} \leq |f(z)| \leq \frac{r}{(1 - qr)^{\frac{1 + q}{1 - q}}} \]

These bounds are sharp because extremal function is the solution of the $q-$ differential equation

\[ \frac{z D_q f(z)}{f(z)} = \frac{1}{1 - qz} \]
Proof. Since the linear transformation \( w = \frac{1}{1 - qz} \) maps \(|z| = r\) onto the disc with the centre \( C(r) = \frac{1}{1 - q^2 r^2} \) and the radius \( \rho(r) = \frac{qr}{1 - q^2 r^2} \). Using theorem 2.1 and subordination principle, then we can write

\[
|D_q f(z) - \frac{1}{1 - q^2 r^2}| \leq \frac{qr}{1 - q^2 r^2}
\]

The inequality (2.7) can be written in the following form

\[
\frac{1}{1 + qr} \leq \Re(e^{i\theta} D_q f(re^{i\theta})) \leq \frac{1}{1 - qr}
\]

On the other hand we have (using the \( q \)-partial rule)

\[
\Re(e^{i\theta} D_q f(re^{i\theta})) = r \frac{\partial_q}{\partial r} \log |f(re^{i\theta})|
\]

Considering (2.8) and (2.9) together we can write

\[
\frac{1}{r(1 + q)} \leq \frac{\partial_q}{\partial r} \log |f(re^{i\theta})| \leq \frac{1}{r(1 - qr)}
\]

If we take \( q \)-integral both sides of (2.10) we get (2.6). \( \square \)

2.6. Remark. Since \( \lim_{q \to 1} \frac{1 - q}{1 - q^2} = 1 \), then (2.6) reduces to

\[
\frac{r}{1 + r} \leq |f(z)| \leq \frac{r}{1 - r}
\]

This is the growth theorem for convex functions [4].

2.7. Theorem. Let \( f(z) \) be an element of \( C_q \), then

\[
(1 + qr)^{-\frac{1 + z^2}{2 \log q - 1}} \leq |D_q f(z)| \leq (1 - qr)^{-\frac{1 + z^2}{2 \log q - 1}}
\]

These bounds are sharp, because extremal function is the solution of the \( q \)-differential equation

\[
1 + qz \frac{D_q f(z)}{D_q f(z)} = \frac{1 + z}{1 - qz}
\]

Proof. Since the linear transformation \( \left( \frac{1 + z}{1 - qz} \right) \) maps \(|z| = r\) onto the disc with the centre \( C(r) = \frac{1 + q^2 r^2}{1 - q^2 r^2} \) and the radius \( \rho(r) = \frac{(1 + q)r}{1 - q^2 r^2} \). Using the definition of \( C_q \) and subordination principle, then we can write

\[
\left| 1 + qz \frac{D_q f(z)}{D_q f(z)} - \frac{1 + q^2 r^2}{1 - q^2 r^2} \right| \leq (1 + qr)
\]

Using the same technique of the proof theorem 2.5, we can obtain

\[
\left| \frac{1 + q}{q} \frac{1 + qr}{1 + q^2} \right| \leq \frac{\partial_q}{\partial r} \log |f(re^{i\theta})| \leq \frac{1 + q}{q} \frac{1}{1 - qr}
\]

Taking \( q \)-integral both sides of (2.12) we get (2.11). \( \square \)

2.8. Remark. Since \( \lim_{q \to 1} \frac{1 - q^2}{q^2 \log q - 1} = 2 \), then (2.11) gives

\[
(1 + r)^{-2} \leq |f'(z)| \leq (1 - r)^{-2}
\]

This is the distortion theorem of convex functions [5].
2.9. Theorem. Let $g(z)$ be an element of $K_q$, then
\begin{equation}
(1 - r)(1 + qr)^{\frac{1 - q^2}{1 - q^2 r^2}} \leq |D_q g(z)| \leq (1 + r)(1 - qr)^{-\frac{1 - q^2}{1 - q^2 r^2}}
\end{equation}
These bounds are sharp because the extremal function is the solution of the $q-$ differential equation
\[\frac{D_q g(z)}{D_q f(z)} = \frac{1 + z}{1 - qz}\]
under the provided that $f(z)$ is $q-$convex.

Proof. Using the definition of the class $K_q$ and theorem 2.1, then we can write
\[\frac{D_q g(z)}{D_q f(z)} = p(z) \Leftrightarrow \frac{D_q g(z)}{D_q f(z)} < \frac{1 + z}{1 - qz} \Rightarrow \frac{D_q g(z)}{D_q f(z)} - \frac{1 + qr^2}{1 - q^2 r^2} \leq (1 + qr)\frac{1}{1 - qr} \Rightarrow \frac{1 - r}{1 + qr} |D_q f(z)| \leq |D_q g(z)| \leq \frac{1 + r}{1 - qr} |D_q f(z)|
\]
which gives (2.13). \hfill \Box

2.10. Theorem. Let $g(z)$ be an element of $K_q$, then
\begin{equation}
\frac{g(z)}{f(z)} \times \frac{1 + z}{1 - qz}
\end{equation}
Proof. Since $g(z) \in K_q$, then we have
\begin{equation}
A(r) = \left\{ \frac{D_q g(z)}{D_q f(z)} : \left| \frac{D_q g(z)}{D_q f(z)} - \frac{1 + q^2}{1 - q^2 r^2} \right| \leq \frac{(1 + qr)}{1 - qr} \right\}
\end{equation}
Now we define the function $\phi(z)$ by
\[\frac{g(z)}{f(z)} = \frac{1 + \phi(z)}{1 - q\phi(z)},\]
thus $\phi(z)$ is analytic and $\phi(0) = 0$. Now we want to show that $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. Assume to the contrary, then there exists $z_0 \in \mathbb{D}$ such that $|\phi(z_0)| = 1$. On the other hand we have
\[\frac{D_q g(z)}{D_q f(z)} = \frac{1 + \phi(z)}{1 - q\phi(z)} \Rightarrow \frac{D_q g(z)}{D_q f(z)} = \frac{g(z)}{f(z)} \frac{1 + q\phi(z)}{(1 - q\phi(z))(1 - q\phi(z))} \frac{f(z)}{D_q f(z)}
\]
or
\begin{equation}
\frac{D_q g(z_0)}{D_q f(z_0)} = \frac{1 + \phi(qz_0)}{1 - q\phi(qz_0)} + \frac{(1 + qz_0)D_q \phi(z_0)}{(1 - q\phi(z_0))(1 - q\phi(qz_0))} \frac{f(z_0)}{D_q f(z_0)}
\end{equation}
In this step, if we use lemma 2.3 (q-Jack’s Lemma) and theorem 2.4, then we have
\[\frac{D_q g(z_0)}{D_q f(z_0)} = \left( \frac{1 + \phi(qz_0)}{1 - q\phi(qz_0)} + \frac{(1 + qz_0)D_q \phi(z_0)}{(1 - q\phi(z_0))(1 - q\phi(qz_0))} \frac{f(z_0)}{D_q f(z_0)} \right) \notin A(r).
\]
But this is a contradiction with (2.15). Therefore $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. We note that the factor $\left( \frac{1 - q^2 r^2}{1 + qr} \right)$ is the reciprocal value of the boundary value of $\left( \frac{z D_q f(z)}{f(z)} \right)$.

2.11. Corollary. If $g(z) \in K_q$, then
\begin{equation}
\left( \frac{r}{1 + qr} \right) \frac{1 - q}{1 - q^2 r^2} \leq |g(z)| \leq \left( \frac{r}{1 + qr} \right) \frac{1 - q}{1 - q^2 r^2}
\end{equation}
Proof. Using the theorem 2.10, then we can write
\[
\frac{1-r}{1+qr} \leq \left| \frac{g(z)}{f(z)} \right| \leq \frac{1+r}{1-qr} \Rightarrow \\
|f(z)| \frac{1-r}{1+qr} \leq |g(z)| \leq |f(z)| \frac{1+r}{1-qr}
\]

In this step, if we use theorem 2.5 we get (2.17) \&

References