ON SOME NEW HADAMARD-TYPE INEQUALITIES FOR CO-ORDINATED QUASI-CONVEX FUNCTIONS

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Abstract
In this paper, we establish some Hadamard-type inequalities based on co-ordinated quasi-convexity. Also we define a new mapping associated with co-ordinated convexity and prove some properties of this mapping.

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1. Introduction

Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a convex function on the interval \( I \) of real numbers and \( a, b \in I \) with \( a < b \). The following double inequality

(1.1) \[ f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2} \]

is well-known in the literature as Hadamard’s inequality. We recall some definitions;

1.1. Definition. (See [4]) A function \( f : [a, b] \to \mathbb{R} \) is said to be quasi-convex on \( [a, b] \) if

\[ f(\lambda x + (1 - \lambda)y) \leq \max \{ f(x), f(y) \}, \quad (QC) \]

holds for all \( x, y \in [a, b] \) and \( \lambda \in [0, 1] \).

Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex. In [1], Dragomir defined convex functions on the co-ordinates as follows:

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1.2. Definition. Let us consider the bidimensional interval $\Delta = [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$. A function $f : \Delta \to \mathbb{R}$ will be called convex on the co-ordinates if the partial mappings $f_x : [a, b] \to \mathbb{R}$, $f_x(u) = f(u, y)$ and $f_y : [c, d] \to \mathbb{R}$, $f_y(v) = f(x, v)$ are convex for all $y \in [c, d]$ and $x \in [a, b]$. Recall that the mapping $f : \Delta \to \mathbb{R}$ is convex on $\Delta$ if the following inequality holds,

\[ f(\lambda x + (1 - \lambda)z, \lambda y + (1 - \lambda)w) \leq \lambda f(x, y) + (1 - \lambda)f(z, w) \]

for all $(x, y), (z, w) \in \Delta$ and $\lambda \in [0, 1]$.

In [1], Dragomir established the following inequalities of Hadamard’s type for co-ordinated convex functions on a rectangle from the plane $\mathbb{R}^2$.

1.3. Theorem. Suppose that $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ is convex on the co-ordinates on $\Delta$. Then one has the inequalities:

\[
\begin{align*}
  f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) & \leq \frac{1}{2} \left[ \frac{1}{b - a} \int_a^b f \left( x, \frac{c + d}{2} \right) \, dx + \frac{1}{d - c} \int_c^d f \left( \frac{a + b}{2}, y \right) \, dy \right] \\
  & \leq \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) \, dy \, dx \\
  & \leq \frac{1}{4} \left[ \frac{1}{b - a} \int_a^b f(x, c) \, dx + \frac{1}{b - a} \int_a^b f(x, d) \, dx + \frac{1}{d - c} \int_c^d f(b, y) \, dy \right] \\
  & \leq \frac{f(a, c) + f(b, c) + f(a, d) + f(b, d)}{4}.
\end{align*}
\]

The above inequalities are sharp.

Similar results for co-ordinated $m$-convex and $(\alpha, m)$-convex functions can be found in [3]. In [1], Dragomir considered a mapping closely connected with the above inequalities and established the main properties of this mapping as follows.

Now, for a mapping $f : \Delta := [a, b] \times [c, d] \to \mathbb{R}$ convex on the co-ordinates on $\Delta$, we can define the mapping $H : [0, 1]^2 \to \mathbb{R}$ by

\[
H(t, s) = \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f \left( tx + (1 - t)\frac{a + b}{2}, sy + (1 - s)\frac{c + d}{2} \right) \, dx \, dy.
\]

1.4. Theorem. Suppose that $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ is convex on the co-ordinates on $\Delta = [a, b] \times [c, d]$. Then:

(i) The mapping $H$ is convex on the co-ordinates on $[0, 1]^2$.

(ii) We have the bounds

\[
\sup_{(t, s) \in [0, 1]^2} H(t, s) = \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) \, dx \, dy = H(1, 1),
\]

\[
\inf_{(t, s) \in [0, 1]^2} H(t, s) = f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) = H(0, 0).
\]

(iii) The mapping $H$ is monotonic nondecreasing on the co-ordinates.

1.5. Definition. Consider a function $f : V \to \mathbb{R}$ defined on a subset $V$ of $\mathbb{R}_n$, $n \in \mathbb{N}$. Let $L = (L_1, L_2, \ldots, L_n)$ where $L_i \geq 0$, $i = 1, 2, \ldots, n$. We say that $f$ is a $L$-Lipschitzian
function if
\[ |f(x) - f(y)| \leq \sum_{i=1}^{n} L |x_i - y_i| \]
for all \( x, y \in V \).

In [2], Özdemir et al. defined quasi-convex function on the co-ordinates as follows:

**1.6. Definition.** A function \( f : \Delta = [a, b] \times [c, d] \to \mathbb{R} \) is said to be a quasi-convex function on the co-ordinates on \( \Delta \) if the following inequality
\[ f(\lambda x + (1 - \lambda) z, \lambda y + (1 - \lambda) w) \leq \max \{ f(x, y), f(z, w) \} \]
holds for all \( (x, y), (z, w) \in \Delta \) and \( \lambda \in [0, 1] \).

Let us consider a bidimensional interval \( \Delta := [a, b] \times [c, d] \). Then \( f : \Delta \to \mathbb{R} \) will be called co-ordinated quasi-convex on the co-ordinates if the partial mappings
\[ f_y : [a, b] \to \mathbb{R}, \ f_y(u) = f(u, y) \]
and
\[ f_z : [c, d] \to \mathbb{R}, \ f_z(v) = f(x, v) \]
are quasi-convex where defined for all \( y \in [c, d] \) and \( x \in [a, b] \). We denote by QC(\( \Delta \)) the class of quasi-convex functions on the co-ordinates on \( \Delta \).

A formal definition of quasi-convex functions on the co-ordinates as follows:

**1.7. Definition.** A function \( f : \Delta = [a, b] \times [c, d] \to \mathbb{R} \) is said to be a quasi-convex function on the co-ordinates on \( \Delta \) if the following inequality
\[ f(\lambda x + (1 - \lambda) y, \lambda u + (1 - \lambda) v) \leq \max \{ f(x, u), f(x, v), f(y, u), f(y, v) \} \]
holds for all \( (x, u), (x, v), (y, u), (y, v) \in \Delta \) and \( \lambda \in [0, 1] \).

In [5], Sarıkaya et al. proved the following Lemma and established some inequalities for co-ordinated convex functions.

**1.8. Lemma.** Let \( f : \Delta = [a, b] \times [c, d] \to \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \frac{\partial^2 f}{\partial t \partial s} \in L(\Delta) \), then the following equality holds:
\begin{align*}
&\frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \ dy \ dx \\
&- \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b [f(x, c) + f(x, d)] \ dx + \frac{1}{d-c} \int_c^d [f(a, y) + f(b, y)] \ dy \right] \\
&= \frac{(b-a)(d-c)}{4} \int_0^1 \int_0^1 (1-2t)(1-2s) \frac{\partial^2 f}{\partial t \partial s} (ta + (1-t)b, sc + (1-s)d) \ dt \ ds.
\end{align*}

The main purpose of this paper is to obtain some inequalities for co-ordinated quasi-convex functions by using Lemma 1.8 and elementary analysis.

### 2. Main Results

**2.1. Theorem.** Let \( f : \Delta \subset \mathbb{R}^2 \to \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \left| \frac{\partial^2 f}{\partial t \partial s} \right| \) is quasi-convex on the co-ordinates on \( \Delta \), then one has the inequality:
\begin{align*}
&\left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \ dy \ dx - A \right| \\
&\leq \frac{(b-a)(d-c)}{16} \max \left\{ \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|, \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|, \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|, \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right| \right\}
\end{align*}
where

\[
A = \frac{1}{2} \left[ \frac{1}{b-a} \int_{a}^{b} [f(x,c) + f(x,d)] \, dx + \frac{1}{d-c} \int_{c}^{d} [f(a,y) + f(b,y)] \, dy \right].
\]

Proof. From Lemma 1.8, we can write

\[
\left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} + \frac{1}{(b-a)(d-c)} \int_{a}^{b} \int_{c}^{d} f(x,y) \, dy \, dx - A \right|
\leq \frac{(b-a)(d-c)}{4} \int_{0}^{1} \int_{0}^{1} \left| (1-2t)(1-2s) \right| \frac{\partial^2 f}{\partial t \partial s} (ta + (1-t)b, sc + (1-s)d) \, dt \, ds.
\]

Since \( \frac{\partial^2 f}{\partial t \partial s} \) is quasi-convex on the co-ordinates on \( \Delta \), we have

\[
\left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} + \frac{1}{(b-a)(d-c)} \int_{a}^{b} \int_{c}^{d} f(x,y) \, dy \, dx - A \right|
\leq \frac{(b-a)(d-c)}{4} \int_{0}^{1} \int_{0}^{1} \left| (1-2t)(1-2s) \right| \left\{ \left| \frac{\partial^2 f}{\partial t \partial s} (a,c) \right|^q, \left| \frac{\partial^2 f}{\partial t \partial s} (a,d) \right|^q, \left| \frac{\partial^2 f}{\partial t \partial s} (b,c) \right|^q, \left| \frac{\partial^2 f}{\partial t \partial s} (b,d) \right|^q \right\} \, dt \, ds.
\]

On the other hand, we have

\[
\int_{0}^{1} \int_{0}^{1} \left| (1-2t)(1-2s) \right| \, dt \, ds = \frac{(b-a)(d-c)}{16}.
\]

The proof is completed. \( \square \)

2.2. Theorem. Let \( f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a,b] \times [c,d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \left| \frac{\partial^2 f}{\partial t \partial s} \right|^q, q > 1 \) is a quasi-convex function on the co-ordinates on \( \Delta \), then one has the inequality:

\[
\left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} + \frac{1}{(b-a)(d-c)} \int_{a}^{b} \int_{c}^{d} f(x,y) \, dy \, dx - A \right|
\leq \frac{(b-a)(d-c)}{4(p+1)^\frac{q}{2}} \left\{ \frac{\partial^2 f}{\partial t \partial s} (a,c), \frac{\partial^2 f}{\partial t \partial s} (a,d), \frac{\partial^2 f}{\partial t \partial s} (b,c), \frac{\partial^2 f}{\partial t \partial s} (b,d) \right\}^\frac{q}{2}
\]

where

\[
A = \frac{1}{2} \left[ \frac{1}{b-a} \int_{a}^{b} [f(x,c) + f(x,d)] \, dx + \frac{1}{d-c} \int_{c}^{d} [f(a,y) + f(b,y)] \, dy \right]
\]

and \( \frac{1}{p} + \frac{1}{q} = 1 \).
Proof. From Lemma 1.8 and using H"older's inequality, we get

\[
\left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) \, dy \, dx - A \right|
\leq \frac{(b-a)(d-c)}{4} \times \int_0^1 \int_0^1 |(1-2t)(1-2s)| \left| \frac{\partial^2 f}{\partial t \partial s} (ta + (1-t)b, sc + (1-s)d) \right| \, dt \, ds
\leq \frac{(b-a)(d-c)}{4} \left( \int_0^1 \int_0^1 |(1-2t)(1-2s)|^p \, dt \, ds \right)^{\frac{1}{p}} \times \left( \int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t \partial s} (ta + (1-t)b, sc + (1-s)d) \right|^q \, dt \, ds \right)^{\frac{1}{q}}.
\]

Since \( |\frac{\partial^2 f}{\partial t \partial s}|^q \) is quasi-convex on the co-ordinates on \( \Delta \), we have

\[
\left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) \, dy \, dx - A \right|
\leq \frac{(b-a)(d-c)}{4} \left( \int_0^1 \int_0^1 |(1-2t)(1-2s)|^p \, dt \, ds \right)^{\frac{1}{p}} \times \left( \max \left\{ \left| \frac{\partial^2 f}{\partial t \partial s} (a,c) \right|^q, \left| \frac{\partial^2 f}{\partial t \partial s} (a,d) \right|^q, \left| \frac{\partial^2 f}{\partial t \partial s} (b,c) \right|^q, \left| \frac{\partial^2 f}{\partial t \partial s} (b,d) \right|^q \right\} \right)^{\frac{1}{q}}.
\]

So, the proof is completed. \( \square \)

2.3. Corollary. Since \( \frac{1}{q} < \frac{1}{(p+1)^{\frac{1}{p}}} < 1 \), for \( p > 1 \) we have the following inequality:

\[
\left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) \, dy \, dx - A \right|
\leq \frac{(b-a)(d-c)}{4} \times \left( \max \left\{ \left| \frac{\partial^2 f}{\partial t \partial s} (a,c) \right|^q, \left| \frac{\partial^2 f}{\partial t \partial s} (a,d) \right|^q, \left| \frac{\partial^2 f}{\partial t \partial s} (b,c) \right|^q, \left| \frac{\partial^2 f}{\partial t \partial s} (b,d) \right|^q \right\} \right)^{\frac{1}{q}}.
\]

2.4. Theorem. Let \( f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a,b] \times [c,d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \left| \frac{\partial^2 f}{\partial x \partial y} \right|^q \), \( q \geq 1 \), is a quasi-convex function on the co-ordinates on \( \Delta \), then one has the inequality:

\[
\left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) \, dy \, dx - A \right|
\leq \frac{(b-a)(d-c)}{16} \times \left( \max \left\{ \left| \frac{\partial^2 f}{\partial t \partial s} (a,c) \right|^q, \left| \frac{\partial^2 f}{\partial t \partial s} (a,d) \right|^q, \left| \frac{\partial^2 f}{\partial t \partial s} (b,c) \right|^q, \left| \frac{\partial^2 f}{\partial t \partial s} (b,d) \right|^q \right\} \right)^{\frac{1}{q}}.
\]
where

\[ A = \frac{1}{2} \left( \frac{1}{b-a} \int_a^b [f(x,c) + f(x,d)] dx + \frac{1}{d-c} \int_c^d [f(a,y) + f(b,y)] dy \right). \]

Proof. From Lemma 1.8 and using the Power Mean inequality, we can write

\[
\left| f(a,c) + f(a,d) + f(b,c) + f(b,d) \right| \leq \frac{(b-a)(d-c)}{4} \left( \int_0^1 \int_0^1 |(1-2t)(1-2s)| \left| \frac{\partial^2 f}{\partial t \partial s} (ta+(1-t)b,sc+(1-s)d) \right| \, dt \, ds \right)^{\frac{1}{q}}
\]

Since \( \frac{\partial^2 f}{\partial t \partial s} \) is quasi-convex on the co-ordinates on \( \Delta \), we have

\[
\left| f(a,c) + f(a,d) + f(b,c) + f(b,d) \right| \leq \frac{(b-a)(d-c)}{4} \left( \int_0^1 \int_0^1 |(1-2t)(1-2s)| \, dt \, ds \right)^{\frac{1}{q}} \left( \int_0^1 \int_0^1 |(1-2t)(1-2s)| \, dt \, ds \right)^{\frac{1}{q}} \left( \max \left\{ \left| \frac{\partial^2 f}{\partial t \partial s} (a,c) \right|^q, \left| \frac{\partial^2 f}{\partial t \partial s} (a,d) \right|^q, \left| \frac{\partial^2 f}{\partial t \partial s} (b,c) \right|^q, \left| \frac{\partial^2 f}{\partial t \partial s} (b,d) \right|^q \right\} \right)^{\frac{1}{q}}
\]

which completes the proof. \( \Box \)

2.5. Remark. Since \( \frac{1}{q} < \frac{1}{(p+1)^\frac{2}{q}} < 1 \) for \( p > 1 \), the estimation in Theorem 2.4 is better than that in Theorem 2.2.

Now, for a mapping \( f : \Delta := [a,b] \times [c,d] \rightarrow \mathbb{R} \) that is convex on the co-ordinates on \( \Delta \), we can define a mapping \( G : [0,1]^2 \rightarrow \mathbb{R} \) by

\[
G(t,s) := \frac{1}{4} \left[ f \left( ta + (1-t) \frac{a+b}{2}, sc + (1-s) \frac{c+d}{2} \right) + f \left( tb + (1-t) \frac{a+b}{2}, sc + (1-s) \frac{c+d}{2} \right) + f \left( ta + (1-t) \frac{a+b}{2}, sd + (1-s) \frac{c+d}{2} \right) + f \left( tb + (1-t) \frac{a+b}{2}, sd + (1-s) \frac{c+d}{2} \right) \right].
\]

We will now give the following theorem which contains some properties of this mapping.
2.6. Theorem. Suppose that \( f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) is convex on the co-ordinates on \( \Delta = [a, b] \times [c, d] \). Then:

(i) The mapping \( G \) is convex on the co-ordinates on \([0, 1]^2\).

(ii) We have the bounds

\[
\inf_{(t, s) \in [0, 1]^2} G(t, s) = \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} = G(0, 0);
\]

\[
\sup_{(t, s) \in [0, 1]^2} G(t, s) = \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} = G(1, 1);
\]

(iii) If \( f \) satisfies the Lipschitzian conditions, then the mapping \( G \) is \( L \)-Lipschitzian on \([0, 1] \times [0, 1]\);

(iv) The following inequality holds:

\[
\frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) \, dy \, dx \leq \frac{1}{4} \left[ f(a, c) + f(b, c) + f(a, d) + f(b, d) + f\left(\frac{a+b}{2}, c\right) + f\left(\frac{a+b}{2}, d\right) + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right].
\]

**Proof.** (i) Let \( s \in [0, 1] \). For all \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \) and \( t_1, t_2 \in [0, 1] \), then we have

\[
G(\alpha t_1 + \beta t_2, s)
\]

\[
= \frac{1}{4} \left[ f\left(\alpha t_1 + \beta t_2)a + (1 - (\alpha t_1 + \beta t_2))a + \frac{b}{2}, sc + (1 - s)\frac{c+d}{2}\right) + f\left((\alpha t_1 + \beta t_2)a + (1 - (\alpha t_1 + \beta t_2))a + \frac{b}{2}, \frac{c+d}{2}\right) + f\left((\alpha t_1 + \beta t_2)a + (1 - (\alpha t_1 + \beta t_2))a + \frac{b}{2}, sc + (1 - s)\frac{c+d}{2}\right) + f\left((\alpha t_1 + \beta t_2)a + (1 - (\alpha t_1 + \beta t_2))a + \frac{b}{2}, \frac{c+d}{2}\right)\right]
\]

\[
= \frac{1}{4} \left[ f\left(\alpha\left(t_1a + (1 - t_1)\frac{a+b}{2}\right) + \beta\left(t_2a + (1 - t_2)\frac{a+b}{2}\right), sc + (1 - s)\frac{c+d}{2}\right) + f\left(\alpha\left(t_1a + (1 - t_1)\frac{a+b}{2}\right) + \beta\left(t_2a + (1 - t_2)\frac{a+b}{2}\right), \frac{c+d}{2}\right) + f\left(\alpha\left(t_1a + (1 - t_1)\frac{a+b}{2}\right) + \beta\left(t_2a + (1 - t_2)\frac{a+b}{2}\right), sc + (1 - s)\frac{c+d}{2}\right) + f\left(\alpha\left(t_1a + (1 - t_1)\frac{a+b}{2}\right) + \beta\left(t_2a + (1 - t_2)\frac{a+b}{2}\right), \frac{c+d}{2}\right)\right].
\]
Using the convexity of $f$, we obtain

$$G(\alpha t_1 + \beta t_2, s) \leq \frac{1}{4} \left[ \alpha \left( \frac{f(t_1 a + (1 - t_1)\frac{a + b}{2}, s c + (1 - s)\frac{c + d}{2})}{f(t_1 b + (1 - t_1)\frac{a + b}{2}, s c + (1 - s)\frac{c + d}{2})} + f(t_1 a + (1 - t_1)\frac{a + b}{2}, s d + (1 - s)\frac{c + d}{2}) + f(t_1 b + (1 - t_1)\frac{a + b}{2}, s d + (1 - s)\frac{c + d}{2}) \right] \right]$$

$$+ \beta \left( \frac{f(t_2 a + (1 - t_2)\frac{a + b}{2}, s c + (1 - s)\frac{c + d}{2})}{f(t_2 b + (1 - t_2)\frac{a + b}{2}, s c + (1 - s)\frac{c + d}{2})} + f(t_2 a + (1 - t_2)\frac{a + b}{2}, s d + (1 - s)\frac{c + d}{2}) + f(t_2 b + (1 - t_2)\frac{a + b}{2}, s d + (1 - s)\frac{c + d}{2}) \right] \right]$$

$$= \alpha G(t_1, s) + \beta G(t_2, s)$$

if $s \in [0, 1]$. For all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $s_1, s_2 \in [0, 1]$, then we also have

$$G(t, \alpha s_1 + \beta s_2) \leq \alpha G(t, s_1) + \beta G(t, s_2),$$

and the statement is proved.

(ii) It is easy to see that by taking $t = s = 0$ and $t = s = 1$, respectively, in $G$, we have the bounds

$$\inf_{(t, s) \in [0, 1]^2} G(t, s) = f\left(\frac{a + b}{2}, \frac{c + d}{2}\right) = G(0, 0),$$

$$\sup_{(t, s) \in [0, 1]^2} G(t, s) = \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} = G(1, 1).$$

(iii) Let $t_1, t_2, s_1, s_2 \in [0, 1]$. Then we have:

$$|G(t_2, s_2) - G(t_1, s_1)|$$

$$= \frac{1}{4} \left| f\left(\frac{t_2 a + (1 - t_2)\frac{a + b}{2}, s_2 c + (1 - s_2)\frac{c + d}{2})}{f(t_2 b + (1 - t_2)\frac{a + b}{2}, s_2 c + (1 - s_2)\frac{c + d}{2})} + f(t_2 a + (1 - t_2)\frac{a + b}{2}, s_2 d + (1 - s_2)\frac{c + d}{2}) + f(t_2 b + (1 - t_2)\frac{a + b}{2}, s_2 d + (1 - s_2)\frac{c + d}{2}) \right|$$

$$+ f\left(\frac{t_2 a + (1 - t_2)\frac{a + b}{2}, s_2 c + (1 - s_2)\frac{c + d}{2})}{f(t_2 b + (1 - t_2)\frac{a + b}{2}, s_2 c + (1 - s_2)\frac{c + d}{2})} + f(t_2 a + (1 - t_2)\frac{a + b}{2}, s_2 d + (1 - s_2)\frac{c + d}{2}) + f(t_2 b + (1 - t_2)\frac{a + b}{2}, s_2 d + (1 - s_2)\frac{c + d}{2}) \right|$$
By using the triangle inequality, we get

\[
\left| G(t_2, s_2) - G(t_1, s_1) \right| \\
\leq \frac{1}{4} \left| f \left( t_2a + (1 - t_2) \frac{a + b}{2}, s_2c + (1 - s_2) \frac{c + d}{2} \right) \right. \\
- f \left( t_1a + (1 - t_1) \frac{a + b}{2}, s_1c + (1 - s_1) \frac{c + d}{2} \right) \\
+ \left| f \left( t_2b + (1 - t_2) \frac{a + b}{2}, s_2c + (1 - s_2) \frac{c + d}{2} \right) \right. \\
- f \left( t_1b + (1 - t_1) \frac{a + b}{2}, s_1c + (1 - s_1) \frac{c + d}{2} \right) \\
- \left| f \left( t_2a + (1 - t_2) \frac{a + b}{2}, s_2d + (1 - s_2) \frac{c + d}{2} \right) \right. \\
- f \left( t_1a + (1 - t_1) \frac{a + b}{2}, s_1d + (1 - s_1) \frac{c + d}{2} \right) \\
+ \left| f \left( t_2b + (1 - t_2) \frac{a + b}{2}, s_2d + (1 - s_2) \frac{c + d}{2} \right) \right. \\
- f \left( t_1b + (1 - t_1) \frac{a + b}{2}, s_1d + (1 - s_1) \frac{c + d}{2} \right) 
\]
\[
\frac{1}{4} |L_1(b-a)|t_2 - t_1| + L_2(d-c)|s_2 - s_1|
+ L_3(b-a)|t_2 - t_1| + L_4(d-c)|s_2 - s_1|
+ L_5(b-a)|t_2 - t_1| + L_6(d-c)|s_2 - s_1|
+ L_7(b-a)|t_2 - t_1| + L_8(d-c)|s_2 - s_1|
\]
\[
= \frac{1}{4} [(L_1 + L_2 + L_3 + L_4)(b-a)|t_2 - t_1|
+ (L_5 + L_6 + L_7 + L_8)(d-c)|s_2 - s_1|]
\]

which implies that the mapping \( G \) is \( L \)-Lipschitzian on \([0, 1] \times [0, 1] \).

(iv) By using the convexity of \( G \) on \([0, 1] \times [0, 1] \), we have
\[
f\left( ta + (1-t)\frac{a+b}{2}, sc + (1-s)\frac{c+d}{2} \right)
+ f\left( tb + (1-t)\frac{a+b}{2}, sc + (1-s)\frac{c+d}{2} \right)
\]
\[
+ f\left( ta + (1-t)\frac{a+b}{2}, sd + (1-s)\frac{c+d}{2} \right)
+ f\left( tb + (1-t)\frac{a+b}{2}, sd + (1-s)\frac{c+d}{2} \right)
\]
\[
\leq tsf(a, c) + t(1-s)f\left( \frac{a+b}{2}, c \right) + (1-t)sf\left( \frac{a+b}{2}, c \right)
+ (1-t)(1-s)f\left( \frac{a+b}{2}, c \right) + tsf(b, c) + t(1-s)f\left( \frac{b+c}{2} \right)
\]
\[
+ (1-t)sf\left( \frac{a+b}{2}, c \right) + (1-t)(1-s)f\left( \frac{a+b}{2}, c \right)
+ tsf(a, d) + t(1-s)f\left( \frac{a+b}{2}, d \right) + (1-t)sf\left( \frac{a+b}{2}, d \right)
\]
\[
+ (1-t)(1-s)f\left( \frac{a+b}{2}, c \right) + tsf(b, d) + t(1-s)f\left( \frac{b+c}{2} \right)
\]
\[
+ (1-t)sf\left( \frac{a+b}{2}, d \right) + (1-t)(1-s)f\left( \frac{a+b}{2}, c \right),
\]

By integrating both sides of the above inequality and taking into account the change of variables, we obtain
\[
\frac{1}{(b-a)(d-c)} \int_{a}^{b} \int_{c}^{d} f(x, y) \, dy \, dx \leq \frac{1}{4} \left[ f(a, c) + f(b, c) + f(a, d) + f(b, d) \right]
\]
\[
+ f\left( \frac{a+b}{2}, c \right) + f\left( \frac{a+b}{2}, d \right).
\]

This completes the proof. \(\square\)

References


