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# Countably mesocompact spaces and function insertion

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**Abstract**

A topological space $X$ is called countably mesocompact if for every countably open cover $\mathcal{U}$ of $X$, there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\{V \in \mathcal{V} : V \cap K \neq \emptyset\}$ is finite for every compact set $K$ in $X$. In this paper, we investigate relations between insertion of semi-continuous functions and a kind of spaces with countable mesocompactness , and give some characterizations of countably mesocompact spaces, $k$-perfect spaces and $k$-MCM spaces.

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**Keywords:** Function insertion, countably mesocompact spaces, $k$-perfect spaces, $k$-MCM spaces.

## 1. Introduction

Let $g$ and $h$ be real valued (non-continuous) functions defined on $X$ and $g \leq h$. Under what conditions does there exist a continuous function $f$ suct that $g \leq f \leq h$? The problem has been investigated extensively. In 1917, Hahn [4] first investigated the particular case in which $g$ is upper semi-continuous and $h$ is lower semi-continuous. In 1951, Dowker[2] proved that a space $X$ is normal and countably paracompact if and only if for each upper semi-continuous function $g : X \to R$ and lower semi-continuous function $h : X \to R$ such that $g < h$, there is a continuous function $f : X \to R$ such that $g < f < h$. In fact, many so called insertion results turn out to provide characterizations of nature and important topological properties such as normal spaces ([5] and [10]), countable paracompact spaces [2], perfectly normal spaces[8] and stratifiable spaces([3] and [6]).

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Boone [1] introduced mesocompactness as a generalization of paracompactness. A topological space $X$ is called (countably) mesocompact, if for every (countably) open cover $\mathcal{U}$ of $X$, there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\{V \in \mathcal{V} : V \cap K \neq \emptyset\}$ is finite for every compact set $K$ in $X$.

Obviously, countable paracompactness $\Rightarrow$ countable mesocompactness $\Rightarrow$ countable metacompactness.

Recently, Xie and Yan [12] discussed the relations between decreasing sequences of sets and insertion of semi-continuous functions and obtained some characterizations of countably metacompact spaces, perfect spaces and countable paracompact spaces.

Enlightened by these results, we present some insertion theorems of countably mesocompact spaces, $k$-perfect spaces and $k$-MCM spaces which are remarkably similar to these characterizations.

Before stating the specific results of this paper, we introduce some notions. The following list of notations seems to be convenient though they may be found in the references: $LSC^+(X)(USC^+(X))$ and $LSC(X)(USC(X))$ represent the sets of all real-valued, strictly positive lower (upper) semi-continuous functions and all real-valued, non-negatively lower (upper) semi-continuous functions on $X$, and we write $UKL(X)(LKU(X))$ for the set of all real-valued, non-negatively upper (lower), and $K$- lower (upper) semi-continuous functions on $X$. By "$g \leq h(g < h)$", We mean $g(x) \leq h(x)(g(x) < h(x))$ for all $x \in X$. $\omega = \mathbb{N} \bigcup \{0\}$ where $\mathbb{N}$ denotes the set of all nature numbers.

A real valued function $f$ defined on a space $X$ is lower (upper) semi-continuous if for any real number $r$, the set $\{x : f(x) > r\}$ (the set $\{x : f(x) < r\}$) is open.

**1.1. Definition** [13]. A real-valued function $f$ defined on a space $X$ is $K$ -lower (upper) semi-continuous if for any compact set $K$, $f$ has a minimum (maximum) value on $K$.

**1.2. Definition** [7]. A space $X$ is said to be $k$-semi-stratifiable, if there is an operator $U$ assigning to each closed set $F$, a sequence $U(F) = (U(j, F))_{j \in N}$ of open sets such that

(1) $F \subseteq U(j, F)$ for each $j \in \mathbb{N}$

(2) if $D \subseteq F$, then $U(j, D) \subseteq U(j, F)$ for each $j \in \mathbb{N}$

(3) $\bigcap_{j \in \mathbb{N}} U(j, F) = F$, and for every compact subset $K$ of $X$, if $K \bigcap F = \phi$, then there exists some $j_0 \in \mathbb{N}$ such that $K \bigcap U(j_0, F) = \emptyset$

Now, we introduce the concept of k-perfect spaces. In fact, $k$-semi-stratifiable spaces can be considered as a monotone version of k-perfect spaces.

**1.3. Definition** . A space $X$ is said to be $k$-perfect, if for each closed subset $F$ of $X$, there exists a decreasing sequence $(U_j)_{j \in \mathbb{N}}$ of open sets such that $F = \bigcap_{j \in \mathbb{N}} U_j$, and for every compact subset $K$ of $X$, if $K \bigcap F = \phi$, then there is some $j_0 \in \mathbb{N}$ such that $K \bigcap U_{j_0} = \phi$.

**1.4. Definition** [11]. A space $X$ is said to be $k$-MCM, if there is a g-function on $X$ such that for every sequence $(x_n)_{n \in \mathbb{N}}$ and compact subset $C$ of $X$, if $g(n, x_n) \bigcap C \neq \emptyset$, then sequence $(x_n)_{n \in \mathbb{N}}$ has a cluster point in $X$.

## 2.Main results

**2.1. Lemma** For any topological space $X$, the following statements are equivalent:

(1) $X$ is countably mesocompact.

(2) If $(U_j)_{j \in \mathbb{N}}$ is an increasing sequence of open sets such that $\bigcup_{j \in \mathbb{N}} U_j = X$, then there exists an increasing sequence $(F_j)_{j \in \mathbb{N}}$ of closed sets such that $\bigcup_{j \in \mathbb{N}} F_j = X$ and $F_j \subseteq U_j$ for each $j \in \mathbb{N}$. Furthermore, for every compact subset $K$ of $X$, there exists some $j_0 \in \mathbb{N}$ such that $K \subseteq F_{j_0}$.

(3) If $(F_j)_{j \in \mathbb{N}}$ is a decreasing sequence of closed sets such that $\bigcap_{j \in N} F_j = \emptyset$, then there exists a decreasing sequence $(U_j)_{j \in \mathbb{N}}$ of open sets such that $\bigcap_{j \in \mathbb{N}} U_j = \emptyset$ and $F_j \subseteq U_j$ for

each $j \in \mathbb{N}$. Furthermore, for every compact subset $K$ of $X$, there exists some $j_0 \in \mathbb{N}$ such that $K \bigcap U_{j_o} = \emptyset$.

*Proof.* (1)⇒(2) Suppose $(U_j)_{j\in\mathbb{N}}$ is a increasing sequence of open sets such that $\bigcup_{j\in\mathbb{N}} U_j = X$, then there exists a compact finite open refinement $\{V_j | j \in \mathbb{N}\}$ of $\{U_j | j \in \mathbb{N}\}$ such that $V_j \subseteq U_j$ for each $j \in \mathbb{N}$ . Let $F_j = X - \bigcup_{k>j} V_k$, then $F_j \subset U_j$ for each $j \in \mathbb{N}$. Thus $(F_j)_{j\in\mathbb{N}}$ is an increasing sequence of closed sets and $\bigcup_{j\in\mathbb{N}} F_j = X$. For every compact subset $K$ of $X$, there exists some $j_0 \in \mathbb{N}$ such that $V_j \bigcap K = \emptyset$ whenever $j > j_o$. So $K \subseteq F_{j_o} = X - \bigcup_{j>j_0} V_j$.

(2)⇒(3) Suppose $(F_j)_{j\in\mathbb{N}}$ is a decreasing sequence of closed sets such that $\bigcap_{j\in\mathbb{N}} F_j = \emptyset$. Let $U_j^{'} = X - F_j$, then $(U_j^{'})_{j\in\mathbb{N}}$ is an increasing sequence of open sets and $\bigcup_{j\in\mathbb{N}} U_j^{'} = X$. Hence there exists an increasing sequence $(F_j^{'})_{j\in\mathbb{N}}$ of closed sets satisfying (2) . Set $U_j = X - F_j^{'}$, then $(U_j)_{j\in\mathbb{N}}$ is a decreasing sequence of open sets and $\bigcap_{j\in\mathbb{N}} U_j = \phi$. One can easily see that $F_j = X - U_j^{'} \subseteq X - F_j^{'} = U_j$. For every compact subset $K$ of $X$, there exists some $j_0 \in N$ such that $K \subseteq F_{j_0}^{'}$, so $K \bigcap U_{j_0} = \phi$.

(3)⇒(1) Suppose $\{G_j | j \in \mathbb{N}\}$ is a countable open covers of $X$. Let $F_j = X - \bigcup_{k\leq j} G_k$, then $(F_j)_{j\in\mathbb{N}}$ is a decreasing sequence of closed sets and $\bigcap_{j\in\mathbb{N}} F_j = \phi$. Thus there exists a decreasing sequence $(U_j)_{j\in\mathbb{N}}$ of open sets satisfying (3). Let $W_1 = G_1, W_2 = U_1 \bigcap G_2, \cdots, W_j = U_{j-1} \bigcap G_j$ for each $j \geq 2$, then$\{W_j | j \in \mathbb{N}\}$ is a open refinement of $\{G_j | j \in \mathbb{N}\}$. For every compact subset $K$ of $X$, there exists some $j_0 \in \mathbb{N}$ such that $K \bigcap U_{j_0} = \phi$, which shows that $K \bigcap W_j = \phi$ whenever $j > j_0$ and $\{W_j | j \in \mathbb{N}\}$ is a compact finite open refinement of $\{G_j | j \in \mathbb{N}\}$. Therefore, $X$ is countably mesocompact.

**2.2. Theorem** For any topological space $X$, the following statements are equivalent:

(1) $X$ is countably mesocompact.

(2) There is an operator $\psi$: $LSC^+(X) \longrightarrow UKL(X)$ such that $0 < \psi(h) < h$ for any $h \in LSC^+(X)$.

(3) There is an operator $\psi$: $USC^+(X) \longrightarrow LKU(X)$ such that $h < \psi(h)$ for each $h \in USC^+(X)$.

*Proof.* (1)⇒(2) Take any $h$ in $LSC^+(X)$. Let $F_j^h = \{x \in X | h(x) \leq \frac{1}{j}\}$ for each $j \in \mathbb{N}$, then one can easily verify that $(F_j^h)_{j\in\mathbb{N}}$ is a decreasing sequence of closed sets and $\bigcap_{j\in\mathbb{N}} F_j^h = \phi$. Thus there exists a decreasing sequence of open sets $(U_j)_{j\in\mathbb{N}}$ satisfying (3) in lemma 2.1. We can define a function $g_h$:$X \longrightarrow R$ by

$$g_h(x) = \begin{cases} 1 & x \in X - U_1 \\ \dfrac{1}{j+1} & x \in U_j - U_{j+1} \end{cases}$$

It is easy to see that $g_h$ is upper semi-continuous. We shall now show that $g_h$ is $K$ lower semi-continuous:

Suppose $K$ is a compact subset of $X$. Let $m$ =min$\{j \in N | K \bigcap U_j = \phi\}$ and $U_0 = X$. For every $x \in K$, there must be a $j^{'} \in \omega$ such that $x \in U_{j^{'}} - U_{j^{'}+1} (j^{'} + 1 \leq m)$ , hence $g_h(x) = \frac{1}{j^{'}+1} \geq \frac{1}{m}$, which implies that $g_h(x)$ has a minimum value $\frac{1}{m}$ on $K$, so $g_h(x)$ is $K$ lower semi-continuous.

Define an operator $\psi$: $LSC^+(X) \longrightarrow UKL(X)$ by $\psi(h) = g_h$. We assert that $0 < \psi(h) < h$ for each $h \in LSC^+(X)$. Take $x$ in $X$. Since $\bigcap_{j\in\mathbb{N}} U_j = \phi$, there exists some $j \in \omega$ such that $x \in U_j - U_{j+1}$, so $\psi(h)(x) = \frac{1}{j+1} < h(x)$ by $x \notin F_{j+1}^h \subseteq U_{j+1}$, which shows that $\psi$ has the required property .

(2)⇒(3) Let $\psi_0$:$LSC^+(X) \longrightarrow UKL(X)$ be an operator satisfying the property in (2). We can define an operator $\psi$; $USC^+(X) \longrightarrow LKU(X)$ by

$$\psi(h) = (\psi_0(\tfrac{1}{h}))^{-1}\ ,\ h \in USC^+(X) \text{ (where } f^{-1} = \tfrac{1}{f}).$$

One can easily verify that the operator $\psi$ satisfies the property in (3).

(3)⇒(1) Let $(U_j)_{j\in\mathbb{N}}$ be an increasing sequence of open sets such that $\bigcup_{j\in\mathbb{N}} U_j = X$, and $\psi$: $USC^+(X) \longrightarrow LKU(X)$ be an operator satisfying the property in (3). Define an upper semi-continuous function $h$: $X \longrightarrow R$ by

$$h(x) = \begin{cases} 1 & x \in U_1 \\ j+1 & x \in U_{j+1} - U_j \end{cases}$$

By hypothesis, $\psi(h)$ is a lower and $K$ upper semi-continuous function such that $h < \psi(h)$. Set $F_j = \{x \in X | \psi(h(x)) \leq j\}$ for each $j \in N$. We assert that $(F_j)_{j\in\mathbb{N}}$ is required. Clearly, $(F_j)_{j\in\mathbb{N}}$ is an increasing sequence of closed sets such that $\bigcup_{j\in N} F_j = X$ and $F_j \subseteq U_j$ for each $j \in \mathbb{N}$. Suppose that $K$ is a compact subset of $X$. Since $\psi(h)$ is a $K$ upper semi-continuous function, there exists $x_0 \in K$ such that $\psi(h(x)) \leq \psi_0(h(x_0))$ for all $x \in K$. Fix $N_0 \in \mathbb{N}$ such that $\psi(h)(x_0) \leq N_0$, then $\psi(h)(x) \leq N_0$ for each $x \in K$, which implies that $K \subseteq F_{N_0}$. By lemma 2.1, $X$ is countably mesocompact.

**2.3. Theorem** For any topological space $X$, the following statements are equivalent:

(1) $X$ is $k$-perfect.

(2) For every open set $U$ of $X$, there exists a increasing sequence $(F_j)_{j\in\mathbb{N}}$ of closed sets such that $U = \bigcup_{j\in\mathbb{N}} F_j$, and for every compact subset $K$ of $X$, if $K \subset U$, then there exists some $j_0 \in \mathbb{N}$ such that $K \subseteq F_{j_0}$.

(3) If $(U_j)_{j\in\mathbb{N}}$ is an increasing sequence of open sets, then there exists an increasing sequence $(F_j)_{j\in\mathbb{N}}$ of closed sets such that $\bigcup_{j\in\mathbb{N}} F_j = \bigcup_{j\in\mathbb{N}} U_j$ and $F_j \subseteq U_j$ for each $j \in \mathbb{N}$, and for every compact subset $K$ of $X$, if $K \subseteq \bigcup_{j\in\mathbb{N}} U_j$ then there exists some $j_0 \in \mathbb{N}$ such that $K \subseteq F_{j_0}$.

(4) If $(F_j)_{j\in\mathbb{N}}$ is a decreasing sequence of closed sets, then there exists a decreasing sequence of open sets $(U_j)_{j\in\mathbb{N}}$ such that $F_j \subseteq U_j$ for each $j \in \mathbb{N}$ and $\bigcap_{j\in N} F_j = \bigcap_{j\in\mathbb{N}} U_j$, and for every compact subset $K$ of $X$, if $K \bigcap F_j = \phi$, then there exists some $j_0 \in \mathbb{N}$ such that $K \bigcap U_{j_0} = \phi$.

(5) There is an operator $\psi$:$LSC(X) \longrightarrow UKL(X)$ such that $0 \leq \psi(h) \leq h$ and $0 < \psi(h)(x) < h(x)$ whenever $h(x) > 0$.

*Proof.* (1)⟺(2) and (3)⟺(4) are clear by de Morgan's Law.

(2)⇒(3) If $(U_j)_{j\in\mathbb{N}}$ is an increasing sequence of open sets. Since $X$ is $k$-perfect, then $U_j = \bigcup_{i\in N} D_{i,j}$ for each $j \in N$, where $(D_{i,j})_{i\in\mathbb{N}}$ is a sequence of closed subsets of $X$ satisfying (2). Let $F_j = \bigcup_{i,i'\leq j} D_{i,i'}$. It is easy to see that $(F_j)_{j\in\mathbb{N}}$ is an increasing sequence of closed sets such that $F_j \subseteq U_j$ for every $j \in \mathbb{N}$ and $\bigcup_{j\in N} F_j = \bigcup_{j\in N} U_j$. For every compact subset $K$ of $X$, if $K \subseteq \bigcup_{j\in\mathbb{N}} U_j$, then there must be some $j_1 \in \mathbb{N}$ such that $K \subseteq U_{j_1}$, thus there is some $i_0 \in \mathbb{N}$ such that $K \subseteq D_{i_0,j_1}$. Let $j_0$=max$\{i_0, j_1\}$, then $K \subseteq F_{j_0}$.

(4)⇒(5) Take any $h$ in $LSC(X)$. Let $F_j^h = \{x \in X | h(x) \leq \frac{1}{j}\}$ for each $j \in \mathbb{N}$, then $(F_j^h)_{j\in\mathbb{N}}$ is a decreasing sequence of closed sets and $\bigcap_{j\in N} F_j = \{x \in X | h(x) = 0\}$, thus there exists a decreasing sequence of open sets $(U_j)_{j\in\mathbb{N}}$ satisfying (4). We can define an upper semi-continuous function $g_h : X \longrightarrow R$ by

$$g_h(x) = \begin{cases} 1 & x \in X - U_1 \\ \dfrac{1}{j+1} & x \in U_j - U_{j+1} \\ 0 & x \in \cap_{j\in\mathbb{N}} U_j \end{cases}$$

.

Now we verify that $g_h$ is $K$-lower semi-continuous:
Let $U_0 = X$. For every compact subset $K$ of $X$, if $K \bigcap U_{j_0} = \phi$ for some $j_0 \in \mathbb{N}$, Set $n_0 = min\{j \in \mathbb{N} | K \bigcap U_j = \phi\}$. Then $K \bigcap U_j = \phi$ for all $j \geq n_0$, and $K \bigcap U_j \neq \phi$ for all $j < n_0$. Take $x_0 \in K \bigcap U_{n_0 - 1}$, then $g_h(x) \geq \frac{1}{n_0} = g_h(x_0)$ for all $x \in K$. If $K \bigcap U_j \neq \phi$ for all $j \in \mathbb{N}$, then $K \bigcap F_j \neq \phi$, thus $K \bigcap (\bigcap_{j\in N} F_j) = K \bigcap (\bigcap_{j\in\mathbb{N}} U_j) \neq \phi$. Take $x_0 \in K \bigcap (\bigcap_{j\in\mathbb{N}} U_j)$, then $g_h(x) \geq 0 = g_h(x_0)$ for all $x \in K$, so $g_h$ is $K$-lower semi-continuous. We only need to show that $0 \leq g_h \leq h$ and $0 < g_h(x) < h(x)$ whenever $h(x) > 0$.

Take $x \in X$. If $h(x) = 0$, then $x \in \bigcap_{j\in\mathbb{N}} F_j = \bigcap_{j\in\mathbb{N}} U_j$, thus $g_h(x) = 0$. If $h(x) > 0$, then $x \notin \bigcap_{j\in\mathbb{N}} F_j = \bigcap_{j\in\mathbb{N}} U_j$. Hence there exists some $j_0 \in \omega$ such that $x \in U_{j_0} - U_{j_0+1}$, thus $g_h(x) = \frac{1}{j_0+1} < h(x)$ by $x \notin U_{j_0+1} \supseteq F_{j_0+1}$. Let an operator $\psi$:$LSC(X) \longrightarrow UKL(X)$ by $\psi(h) = g_h$, then $\psi$ has the required property .

(5)$\Rightarrow$(2) Let $U$ be an open subset of $X$ , and $\psi$: $LSC(X) \longrightarrow UKL(X)$ be an operator satisfying the property in (5). We can define a lower semi-continuous function $h_U : X \longrightarrow R$ by

$$h_U(x) = \begin{cases} 1 & x \in U \\ 0 & x \in X - U \end{cases}$$

.

Then $h_U \in LSC(X)$, so $\psi(h_U)$ is an upper and $K$- lower semi-continuous function such that $0 \leq \psi(h_U) \leq h_U$ and $0 < \psi(h_U)(x) < h_U(x)$ whenever $h_U(x) > 0$. By putting $F_n = \{x : \psi(h_U) \geq \frac{1}{n}\}$, we obtain an increasing sequence $\{F_n : n \in N\}$ of closed subset of $X$. It is easy to check that $\{F_n : n \in \mathbb{N}\}$ satisfies (2).

As a monotone version of $k$-perfects spaces, $k$-semi-stratifiable spaces have been characterized by monotone insertion of functions [13]. Naturally, we hope to find the monotone version of countably mesocompact spaces and give its characterizations. Peng and Lin[8] gave $k$-MCM spaces characterizations as following. In fact, $k$-MCM spaces can be seen a monotone version of countably mesocompact spaces.

**2.4. Theorem [8]** For any topological space $X$,the following statements are equivalent:

(1) $X$ is $k - MCM$.

(2) There exists an operator $U$, assigning to every decreasing sequence $\{D_j\}_{j\in\mathbb{N}}$ of closed sets with empty intersection, a sequence $U(\{D_j\}) = \{U(n, \{D_j\})\}_{n\in\mathbb{N}}$ of open sets such that:

(i) $D_n \subset U(n, \{D_j\})\}$ for each $n \in \mathbb{N}$;

(ii) For every compact subset $C$ of $X$, there exists some $m \in \mathbb{N}$, such that $U(m, \{D_j\}) \bigcap C = \phi$;

(iii) If $D_n \subseteq E_n$, then $U(n, \{D_j\}) \subseteq U(n, \{E_j\})$.

**2.5. Theorem** For any topological space $X$, the following statements are equivalent:

(1) $X$ is $k - MCM$.

(2) There is an operator $\psi$:$LSC^+(X) \longrightarrow UKL(X)$ such that, for any $h \in LSC^+(X), 0 < \psi(h) < h; \psi(h_1) \leq \psi(h_2)$ whenever $h_1 \leq h_2$.

(3) There is an operator $\psi$:$USC^+(X) \longrightarrow LKU(X)$ such that for any $h \in USC^+(X), h < \psi(h), \psi(h_1) \leq \psi(h_2)$ whenever $h_1 \leq h_2$.

*Proof.* (1)$\Rightarrow$(2) Take any $h \in LSC^+(X)$. Let $F_j^h = \{x \in X | h(x) \leq \frac{1}{j}\}$ for each $j \in \mathbb{N}$, then $(F_j^h)_{j \in \mathbb{N}}$ is a decreasing sequence of closed sets and $\bigcap_{j \in N} F_j^h = \phi$. So there exists a decreasing sequence $(U(n, (F_j^h)))_{n \in \mathbb{N}} = U((F_j^h)_{j \in \mathbb{N}})$ of open subsets of $X$ satisfying (2) in theorem 2.4. Define an upper semi-continuous function $\psi(h)$:$X \longrightarrow R$ by

$$\psi(h)(x) = \begin{cases} 1 & x \in X - U(1, (F_j^h)) \\ \dfrac{1}{n+1} & x \in U(n, (F_j^h)) - U(n+1, (F_j^h)) \end{cases}$$

.

We assert that the operator $\psi(h)$ has the required property in (2). By the proof of (1)$\Rightarrow$(2) in Theorem 2.2, it is obvious that $\psi(h)$ is $K$- lower semi-continuous on $X$. We only need to show that $\psi(h)$ is an order-preserving operator. If $h_1 \leq h_2$, clearly, $F_j^{h_1} \supseteq F_j^{h_2}$ for each $j \in \mathbb{N}$, where $F_j^{h_i} = \{x \in X | h_i(x) \leq \frac{1}{j}\} (i = 1, 2)$. By hypothesis, $U(n, (F_j^{h_1})) \supseteq U(n, (F_j^{h_2}))$ for each $n \in \mathbb{N}$, where $(U(n, (F_j^{h_i})))_{n \in N} = U((F_j^{h_i})_{j \in \mathbb{N}}), (i = 1, 2)$. Take $x \in X$, then there exists $n_0 \in \mathbb{N}$ such that $x \in U(n_0, (F_j^{h_1})) - U(n_0+1, (F_j^{h_1}))$ (let $U(0, (F_j^{h_1}))) = X$), by $\bigcap_{n \in \mathbb{N}} U(n, (F_j^{h_1})) = \phi$. Furthermore, $x \notin U(n_0 + 1, (F_j^{h_1})) \supseteq U(n_0 + 1, (F_j^{h_2}))$. This implies that $\psi(h_1)(x) = \frac{1}{n_0+1} \leq \psi(h_2)(x)$. Therefore, $\psi(h_1) \leq \psi(h_2)$.

(2)$\Rightarrow$(3) Let $\psi_0 : LSC^+(X) \longrightarrow UKL(X)$ be an operator satisfying the property in (2). We can define an operator $\psi : USC^+(X) \longrightarrow LKU(X)$ by $\psi(h) = (\psi_0(h^{-1}))^{-1}, h \in USC^+(X)$. (Where $f^{-1} = \frac{1}{f}$). One can easily verify that the operator $\psi$ satisfying the property in (3):

(3)$\Rightarrow$(1) Let $\psi$ be an operator having the property in (3). For each decreasing sequence of closed sets $(D_j)_{j \in \mathbb{N}}$ in $X$ such that $\bigcap_{j \in \mathbb{N}} D_j = \phi$, we can define an upper semi-continuous function $h_{(D_j)} : X \to \mathbb{R}$ by

$$h_{(D_j)}(x) = \begin{cases} 1 & x \in X - D_1 \\ n+1 & x \in D_n - D_{n+1}. \end{cases} \qquad (*)$$

Then $\psi(h_{(D_j)})$ is a lower and K upper semi-continuous function such that $h_{(D_j)} < \psi(h_{(D_j)}$ by (3). Define an operator $U$ assigning to each decreasing sequence of closed sets $(D_j)_{j \in \mathbb{N}}$, a decreasing sequence of open sets with empty intersection by $U((D_j)) = (U(n, (D_j)))_{n \in \mathbb{N}}$ where $U(n, (D_j)) = \{x \in X \mid \psi(h_{(D_j)})(x) > n\}$ for each $n \in \mathbb{N}$. We assert that the operator $U$ satisfies (2) in theorem 2.4. In fact, one can easily verify that (i),(ii) in (2) hold for $U$. Thus we only need to show that $U$ satisfies (iii) in (2). Take two decreasing sequences of closed sets $(F_j)_{j \in N}$ and $(E_j)_{j \in \mathbb{N}}$ such that $F_j \subseteq E_j$ for each $j \in \mathbb{N}$, and $\bigcap_{j \in N} F_j = \bigcap_{j \in \mathbb{N}} E_j = \phi$. Then one can easily obtain $h_{(F_j)} \leq h_{(E_j)}$, where $h_{(F_j)}$ and $h_{(E_j)}$ are defined by($*$), thus $\psi(h_{(F_j))} \leq \psi(h_{(E_j))}$ by $\psi$ being order-preserving. Hence, $U(n, (F_j)) = \{x \in X | \psi(h_{(F_j))}(x) > n\} \subseteq \{x \in X | \psi(h_{(E_j))}(x) > n\} = U(n, (E_j))$ for each $n \in \mathbb{N}$. This shows that (iii) in theorem 2.4 holds for $U$.

## References


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